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Estate Division Problem: The Core and The Godly Interference

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Abstract. This article offers a simple method to reduce the Core of a Transferable Utility game to a singleton where the Core happens to have more than one elements in it. By presenting a popular folklore, this article infers that the solution of an integer-constrained estate division problem can be achieved by introducing a temporary player who gets no utility by participating in the game. This article refers to this temporary player as a god. It shows that such 'god'-ly interference can help us to make sense the Core by making it a unique one.

One day, a farmer dies in a distant village (see [7] and Problem 172 in [3]). He is survived by his 3 children and 17 cows. While dying, he conveys his last wish to his children. As loyal his children are, they agree to respect their father's wish. The farmer says— "I want my eldest child to have half of all my cows. One third of all my cows goes to the second child. And the youngest one will take 1/9th of all 17 cows. But you remember that, while sharing you must make sure that you do not kill any of the cows." Upon agreeing to their father's last wish, now his children find themselves in a riddle as they have to solve a very difficult mathematical problem in order to divide 17 cows according to the farmer's wish without killing one, After many failed attempts, they finally arrive to the village chief's house and asks for help to solve the riddle. The chief carefully listens to the problem and decides to call a 4th person in the meeting. The chief tells the farmer's children— "As the proportions of the number of your father's cows does not add up to 1, I propose that this 4th person brings some more cows, donates those cows in your father's name, and claims 1/18th of the final number of cows owned by your father". It is a strange proposal. Everybody is confused. In the meantime, a wise person of the village arrives at the spot, and whispers to the 4th person— "just buy one cow and all will agree". The wise person leaves and the story ends.

1. INTRODUCTION Problems related to divisions of an estate have been amply present since antiquity. One of the oldest of such problems is mentioned in the Talmud, the Jewish religious scripture. For this particular problem, the Talmud prescribes a peculiar solution that suggests a certain way of dividing an estate where the total claim on the estate by the claimants is more than the worth of the estate. Aumann and Maschler [1] have demonstrated a game theoretic interpretation to this problem. Their interpretation, consequently, has augmented prolific research in the domain of conflicting claims problems, the theoretical term used for the estate division problems. The formal analysis of these problems is introduced in a seminal paper by O'Neill [9]. Subsequent research shows that a crucial number of well-behaved solutions exist for solving conflicting claims problems. Concepts such as the 'proportional', the 'constrained equal awards', the 'constrained equal losses', the 'Talmud' and the 'random arrival rules' are used vividly in solving the conflicting claims problems.

The folktale we use in this paper is an example where the total claim to an estate is smaller than the worth of the estate, contrary to the conflicting claims problems. Within the domain of finance, this problem is frequently solved using the rule of proportional division. However, if one emphasizes on fairness and stability of an agreement, the rule of proportional division is not the best way to divide a pie. This essay indicates how to analyse such kind of problems in order to arrive at a stable solution. Another important point in our folktale is that the estate can be divided only in integer quantities, and non-integer division is not allowed as per the rule of division(see [4, 5] as examples of literature dealing with integer divisions in cooperative games). A situation where the total claim to an estate is lesser than the estate itself is at the heart of this article that presents a game theoretic interpretation of the problem using the folklore.

2. BUILDING A MODEL

Setting up the Question We formulate our model as a cooperative game. Recall that a cooperative game is given by (N, v), where N denotes the set of players and $v : 2^N \to \mathbb{R}$ is the characteristic function. For a subset $S \subseteq N$ of players, $v(S)$ is the worth of the coalition S. For more details about cooperative games, we refer to [8].

In the cooperative game pertaining to the problem at hand, we assume that the 4th person buys x cows. Our task is to find out what should be the value of x so that everyone will agree to the proposal made by the village chief. For everyone to agree to the chief's proposal, there is one mandatory requirement that needs to be satisfied which is: Nobody should be able to benefit by creating a smaller subgroup out of the group of 4 players that are the farmer's 3 children and the 4th person. Let us analyze the game mathematically.

All 4 players have the right to agree or disagree to the village chief's proposal. If they agree, then well and good for everyone. If they disagree, they get their share as commanded by their father up to the largest integer. It means that, if the eldest child disagrees to the proposal, that person will get the integral part of $(1/2) * 17 = 8$ cows. cows. similarly, it can be calculated what happens when each of the four players disagrees to the proposal. We will denote the outcome as a function v_M of a division game M .

We will define two such games where in one game only the farmer's children will be the player and in the other game, the 4th person along with farmer's 3 children will play the game. In each game, there can be two variants based on two different kinds of characteristic functions. In one characteristic function, for example, if the eldest child and the second child decides to form a group and deal about the division as a unit, then they can get either $\left\lfloor \frac{17}{2} \right\rfloor + \left\lfloor \frac{17}{3} \right\rfloor = 14$ cows or they can get $\left\lfloor \frac{17}{2} + \frac{17}{3} \right\rfloor = 15$ cows (Here $\lfloor y \rfloor$ denotes the largest integer less than or equal to y). We will denote the first version of the 3-player game as the game M_1, and the second as M_2. Similarly, we will denote the first version of the 4-player game as the game T_1, and the second as T_2. The characteristic functions of these 4 games are as follows :

$$v_{M_1}(1) = 8,\ v_{M_1}(2) = 5,\ v_{M_1}(3) = 1,\ v_{M_1}(12) = 13,\ v_{M_1}(13) = 9,\ v_{M_1}(23) = 6,\ v_{M_1}(123) = 17$$

$$v_{M_2}(1) = 8,\ v_{M_2}(2) = 5,\ v_{M_2}(3) = 1,\ v_{M_2}(12) = 14,\ v_{M_2}(13) = 10,\ v_{M_2}(23) = 7,\ v_{M_2}(123) = 17$$

$$v_{T_1}(1) = \left\lfloor \frac{17+x}{2} \right\rfloor = v_{T_1}(14),\ v_{T_1}(2) = \left\lfloor \frac{17+x}{3} \right\rfloor = v_{T_1}(24),\ v_{T_1}(3) = \left\lfloor \frac{17+x}{9} \right\rfloor = v_{T_1}(34),\ v_{T_1}(4) = x,$$

$$v_{T_1}(i \cup j) = v_{T_1}(i) + v_{T_1}(j),\ v_{T_1}(1234) = 17 + x$$

$$v_{T_2}(1) = \left\lfloor \frac{17+x}{2} \right\rfloor = v_{T_2}(14),\ v_{T_2}(2) = \left\lfloor \frac{17+x}{3} \right\rfloor = v_{T_2}(24),\ v_{T_2}(3) = \left\lfloor \frac{17+x}{9} \right\rfloor = v_{T_2}(34),\ v_{T_2}(4) = x,$$

$$v_{T_2}(i \cup j) = \lfloor (i's\ proportion + j's\ proportion) \rfloor,\ v_{T_2}(1234) = 17 + x$$

In order to improve readability, for the games T_i, we have omitted the explicit expressions of 3-player coalition values and put them as formulas. Also, for the games T_i, any single player's coalition value is same as the coalition value of that player with the 4th player because the 4th player does not bring anything or get anything unless they all agree to the chief's proposal, that is, the grand coalition.

The games M_1 & M_2, within the scope of cooperative game theory, have many stable solutions in their Core. When we say stable solutions here, we refer to such solutions where no one gets less in the grand coalition than what they can get if they do not agree to play along. Such an agreement can be $(8, 7, 2)$ where nobody has any incentive to move out of the coalition and go alone. Another such agreement that can be reached upon is $(9, 6, 2)$. In a similar way, we can find many more such agreeable solutions. For there being so many elements in the Core of this game, it is a difficult job to make sense of the Core. We can calculate the other popular solution concepts such as Shapley Value and Nucleolus of the games M_1 & M_2 and in both the cases the Shapley Value and the Nucleolus turns out to be the same allocation of $(9, 6, 2)$. However, neither Shapley Value, nor Nucleolus talks about stability of agreement for the proposed allocation, and hence we turn our attention back to Core. It is important to note here that we are emphasizing on stability in this article because the central tenet of the folk tale is agreement to a certain division. Now if we are to predict an agreement, or to devise a policy to make people agree to an agreement, a stable agreement is something that is desirable in those cases. Hence, we should look for the Core in the second set of games T_1 & T_2 and try to make sense of the Core.

Towards the Answer When the chief proposed the inclusion of the 4th person, the original game transforms to the games T_1 and T_2. This means that they are left with $17 + x$ cows. As per the rule of inheritance, in either games if they all agree to the chief's proposal, the eldest child gets $v_{T_i}(1) = \left\lfloor \frac{17+x}{2} \right\rfloor$ cows, the second child gets $v_{T_i}(2) = \left\lfloor \frac{17+x}{3} \right\rfloor$ cows, the youngest one gets $v_{T_i}(3) = \left\lfloor \frac{17+x}{9} \right\rfloor$ cows, and the 4th person gets $v_{T_i}(4) = \left\lfloor \frac{17+x}{18} \right\rfloor - x$ cows, because the 4th person had to contribute x cows to the total number of cows beforehand.

The question for us to answer now is: What should be the value of x so that none of the four players has a reason to disagree? That value of x must be such that, in the former situation, where they disagreed to include the 4th person in the game, everyone's share must not be more than that of the later situation where they agree to the chief's proposal. Hence, x should satisfy all the following inequalities::

$$\begin{cases} v_{T_i}(1) \geq v_{M_i}(1) \\ v_{T_i}(2) \geq v_{M_i}(2) \\ v_{T_i}(3) \geq v_{M_i}(3) \\ v_{T_i}(4) \geq 0 \\ \quad x \geq 1 \end{cases} \tag{2.1}$$

Any violation of the above inequalities must point to an impossibility of an agreement.

The equations above implies:

$$\begin{cases} \dfrac{17+x}{2} \geq 8 \\[4pt] \dfrac{17+x}{3} \geq 5 \\[4pt] \dfrac{17+x}{9} \geq 1 \\[4pt] \dfrac{17+x}{18} \geq x \\[4pt] x \geq 1 \end{cases}$$

And not so surprisingly, the only value of x that satisfies all these inequalities is $x = 1$. The wise person seems really wise now. Since $x = 1$, nobody has any unilateral incentive to disagree to the proposal as they cannot get more by disagreeing. We have, so far, solved the situation where no player disagrees alone. However, one question remains: What about disagreeing along with someone as a group?

This question, although looks complicated, can easily be solved using the same equations formulated above. As any group G which does not include the 4th person gets exactly the summation of their individual $v(i),\ i \in G$, that cannot be better than the situation where $x = 1$, as we just proved. On the other hand, in any dissenting group G_1 which has the 4th person in it, each player gets the same payoff as they would have gotten in the grand coalition. So the dissenting coalition does not also do them any better.

Therefore, the solution that we have described above is the unique Core of the cooperative games T_1 & T_2. It is important, here, to note that the same allocation for the farmer's 3 children is also in the Core of M_1 & M_2. The wise person really knows his game theory well, as it turns out! The farmer's children get their fair share as per their father's wish, and the 4th person gets the investment back. Moreover, no one has any reason to disagree to this proposal. This takes us to a more challenging problem in Cooperative Game Theory. we discuss this in the following section.

3. DISCUSSION Though Core is a widely accepted solution concept in game theory, the problem with the concept is that it does not have a guaranteed and/or unique existence. For example, no essential n-person zero-sum game has a non-empty Core and that constitutes a plethora of cooperative games [6]. For other games, the Bondareva-Shapley theorem [10] states that any game will have a non-empty core, if and only if the game is balanced. Thus, the concept of Core, even though inviting for its inherent idea of stability, has not been proven to be a much useful solution concept in the domain of Game Theory. Often, even when the Core exists, due to its multiplicity, it becomes difficult to make sense of the Core. Although the games M_1 & M_2 has the same allocation, that is in the unique Core of T_1 & T_2 $(9,6,2)$, in their Core, there are several other elements in M_i's Core as well. Unfortunately, in this case the Core does not make sense for either of M_1 & M_2. Stability becomes challenging to decipher when there are multiple stable alternatives. The idea of stability becomes questionable under such alternatives. What we have tried to offer in this paper is to show that when $x = 1$ in the modified game T_i, the Core of the original game M_i, at once, makes sense since it is the unique Core now. In other words, in presenting a model of a popular folklore, we have demonstrated in this article how we can make sense of the Core by making it unique by inducing a certain kind of perturbation on the game. An inclusion of a transient player, in our curated model of the folklore, conclusively leads to making sense of a Core in the game where the Core was not making sense in the first place.

Although we change the game by adding a 4th player and solve the game thusly, the solution constituted by the inequalities 2.1 represents individual rationality condition in the original 3-person game. In addition, since the characteristic function v of the game is supermodular, the collective and coallitional rationality conditions for the original 3-person game are also satisfied.

Therefore, in essence, this article offers the inference that it is possible for the Core to make sense. In other words, it is possible to have a unique Core if we introduce some kind of transient perturbation in games where the Core is non-singleton.

4. MAY THE 4TH BE WITH US In a manner, the 4th person in our model acts like the 'law'. We say this in the sense of Basu [2], when he discusses how law can change people's behaviour without gaining anything from the game: "The law creates a new focal point and this [. . .] influences behavior". As we have known since the beginning of time, law is god's will. The 4th person, in this article, acts like a harbinger of focal point in the game we have composed. Without contaminating the game with the 4th person, it is not, otherwise, possible to arrive at a solution of a unique Core because of the multiplicity of the Core. We submit this type of perturbation as the 4th person does to the game as 'god'ly interference, as the 4th person happens to appear almost as a god who gets no payoff from participating in the game but solves the unsolvable problem and leaves the arena.

We cannot, at this point, be certain whether every game with a non-unique Core can be handled in the same manner that we have presented or not. We leave this possibility open to explore more in future. However, the fact that it is possible to have a unique Core in this particular game in concern points us towards the feasibility of making sense of the Core in an erstwhile impossible situation. We are pleased to show a novel route to reach the elusive solution concept of a unique Core in such games where the Core is non-singleton by creating simple temporary disturbances in the structure.

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