INDIAN INSTITUTE OF MANAGEMENT KOZHIKODE

Working Paper

IIMK/WPS/349/QM&OM/2019/08

November 2019

Optimum accelerated life test sampling plans for Type-I hybrid censored Weibull distributed products sold under general rebate warranty

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Abstract: In order to reduce avoidably lengthy duration required to test highly reliable products under usage stress, accelerated life test sampling plans are employed. In this paper, accelerated life test sampling plans (ALTSP) are developed for Type-I hybrid censored products sold under the general rebate warranty. The primary decision model proposed in this paper determines ALTSP by minimizing the relevant costs involved. The optimal solution is attained by utilizing appropriate analysis techniques following a constrained optimization approach. As a special case, ALTSP for Type-I censoring is obtained using the same approach. ALTSP under Type-I hybrid censoring using the variance minimization method is also derived. A well-designed sensitivity analysis is incorporated using a real-life failure dataset pertaining to low-metallic break-pads to analyze the sensitivity of the optimal solution due to the mis-specification of parameter values. The model is exemplified using a real-life case.

Keywords: Accelerated life tests, Sampling plans, Weibull distribution, Type-I hybrid censoring, General rebate warranty, Constrained optimization.

1 Introduction

As an industrial quality control technique, acceptance sampling is commonly used in production units. According to Wu *et al* (2015) acceptance sampling plans reduces the gap between the actual and expected quality of manufactured goods. For products with lifetime as an important quality characteristic, life test sampling plan is a widely accepted technique to decide the acceptability of a lot. Since the response values are not observable for all the units under study, hence lifetime data are typically censored. The two most commonly utilized censoring schemes are Type-I censoring (time-censoring) scheme and Type-II censoring (failure-censoring) scheme. In Type-I censoring, the test is aborted after a pre-decided time x_0; whereas in Type-II censoring, the termination of the test is subject to failure of a pre-fixed number of items r.

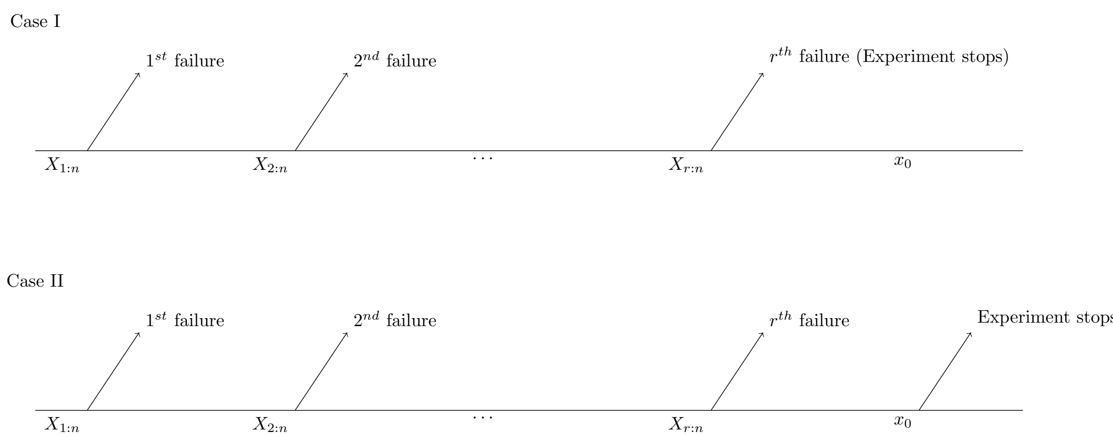
Life test sampling plans using the two aforementioned censoring schemes and their extensions have been studied extensively in the literature. Schneider (1989) developed Type-II censored life test sampling plans for products following lognormal and Weibull distributions. Yeh (1994) followed a Bayesian approach to develop

Type-I censored life test sampling plans for products having exponentially distributed lifetimes. Similarly, Kockerlakota and Balakrishnan (1986), Balasooriya (1995), Balasooriya and Saw (1998), Balasooriya *et al.* (2000) etc. have developed life test sampling plans using different combinations of methods, lifetime distributions, and censoring schemes.

Epstein (1954) introduced hybrid censoring scheme by combining Type-I and Type-II censoring schemes for the first time. Eventually, the censoring scheme introduced by Epstein (1954) came to be known as Type-I hybrid censoring scheme. It can be described briefly as follows: Let us consider n identical units are put on test. Now if $X_{1:n}, ..., X_{n:n}$ be the ordered lifetimes of the units put on test, then the experiment is aborted either when a pre-chosen number $r < n$ out of n items has failed or when a pre-determined time x_0 has elapsed. Hence the life test can be terminated at a random time $X^* = min\{X_{r:n}, x_0\}$. One of the following two types of observations can be witnessed under Type-I hybrid censoring scheme.

Case I: $\{X_{1:n} < ... < X_{r:n}\}$ if $X_{r:n} < x_0$.
Case II: $\{X_{1:n} < ... < X_{d:n} < X_0\}$ if $d + 1 \leq r < n$ and $x_0 \leq X_{r:n}$.

Figure 1: Schematic illustration of Type-I hybrid censoring scheme



Type-I hybrid censoring schemes have been considered by several authors to develop life test sampling plans at usage stress. Dube *et al* (2011) developed Type-I hybrid censored sampling plans for products with lifetimes following lognormal distribution. Subsequently, Bhattacharya *et al* (2014) and Bhattacharya *et al* (2015) used Type-I hybrid censoring scheme to develop life test sampling plans for products with Weibull distributed lifetimes by optimizing cost and asymptotic variance respectively.

Due to advancements in technology and processes, modern-day products are highly reliable. To test lifetimes of such products at usage stress is a taxing lengthy practice. Hence, in order to overcome this hindrance, accelerated life testing is employed to develop life test sampling plans. Several authors have used accelerated life tests for developing life test sampling plans using Type-I, Type-II and other variants of the two censoring schemes. Yum and Kim (1990) developed failure censored accelerated life test sampling plan (ALTSP) for products with exponentially distributed lifetimes. As an extension of Yum and Kim (1990), Hsieh (1994) developed ALTSPs which minimized the total number of failures at each stress levels. Bai *et al* (1993) also obtained ALTSP using Type-II censoring for lognormal and Weibull distributions. Wang (1991) introduced ALTSP using Type-I censoring which allowed zero failures during the test. Ke (1999) and later Lu and Rudy (2001) using a similar setup as Wang (1991) developed ALTSPs which allowed more than zero failures. Seo *et al* (2009) introduced ALTSPs using both Type-I and Type-II censoring for non-constant shape parameter. So far, only Kim and Yum (2011) developed ALTSP using Type-I hybrid censoring under the assumption that acceleration factor between the accelerated and used conditions and the shape parameter are known.

The study of existing literature reveals that even if several papers developed life test sampling plans (LTSP) under accelerated life test setting, cost considerations have seldom been made in the process. For a consumer durable product warranty cost is an indispensable cost. Several papers (Kwon, 1996; Huang *et al*, 2008; Tsai *et al* 2008; Hsieh and Lu, 2013 etc) under usage stress have acknowledged the importance of warranty cost while designing LTSP. But to the best of our knowledge, no paper so far has included warranty cost while designing LTSPs under accelerated life test setting. In this paper, we determine ALTSPs in presence of Type-I hybrid censoring using a cost function approach for products sold under the general rebate warranty scheme having Weibull lifetimes. ALTSPs for products with Weibull lifetimes in the presence of Type-I hybrid censoring using asymptotic variance minimization approach has also been obtained. Evidence from the literature shows that ALTSPs using both the aforementioned approaches have not been studied for Type-I hybrid censoring scheme. We inculcate a constrained optimization approach to account for producer's and consumer's risk in determining the ALTSPs. The rest of the paper is organized as follows. In Section 2 we discuss in detail the model framework. We describe the relevant costs involved and formulate the expected cost minimization problem in Section 3. In Section 4, an alternate approach of obtaining ALTSPs using asymptotic variance minimization is discussed. A well-designed sensitivity analysis is conducted in Section 5 by introducing a real-life failure dataset pertaining to low-metallic brake pads. A real life case is illustrated in Section 6. Finally, we put down our conclusion in Section 7.

2 Model framework

2.1 Weibull distributed lifetime

Let the lifetime X of a testing unit follow Weibull distribution at stress s_i with probability density function (pdf), $f_X(x)$ given by

$$f_X(x) = k{\lambda_i}^k x^{k-1} e^{-(\lambda_i x)^k}; x > 0, \tag{2.1}$$

where $k > 0$ is the constant shape parameter and $\lambda_i > 0$ is the scale parameter which can be represented using the following relationship with stress s_i

$$\ln(\lambda_i) = \kappa_0 + \kappa_1 g(s_i). \tag{2.2}$$

The stress translation function in (2.2) can assume Arrhenius, log-linear, power rule and other relationships depending on the kind of stress used for testing. For more details refer to Mukhopadhyay and Roy (2016). The corresponding cumulative distribution function (CDF), $F_X(x)$ can be written as

$$F_X(x) = 1 - e^{-(\lambda_i x)^k}; x > 0. \tag{2.3}$$

If we consider the transformation $T = \ln X$, the corresponding CDF of the of the extreme value distribution of T is given by

$$F_T(t) = 1 - e^{-e^{\frac{t-\mu_i}{\sigma}}}; -\infty < t < \infty, \tag{2.4}$$

where $-\infty < \mu_i < \infty$ and $\sigma > 0$ are the respective location and scale parameters given by $\mu_i = -\ln \lambda_i$ and $\sigma = \frac{1}{k}$. The location parameter can be expressed using the standardized stress ξ_i as

$$\mu_i = \gamma_0 + \gamma_1 \xi_i; \tag{2.5}$$

where, $\xi_i = \frac{g(s_i)-g(s_0)}{g(s_H)-g(s_0)}$, $\gamma_0 = -(\kappa_0 + \kappa_1 g(s_0))$, and $\gamma_1 = -\kappa_1(g(s_H) - g(s_0))$. s_0 and s_H are used condition stress and highest stress respectively. Therefore, $\xi_i = 0$ when $s_i = s_0$ and $\xi_i = 1$ when $s_i = s_H$. Also, when $\xi_i = 0$, $\mu_0 = \gamma_0$.

2.2 Fisher information matrix

Let $X_1, X_2, ..., X_{n_i}$ be the lifetimes of n_i (number of items to be tested for i^{th} stress level) units to be put on test at stress level s_i which follow Weibull distribution given by (2.3). Hence, $T_1, T_2, ..., T_{n_i}$ will be the corresponding log-lifetimes which follow extreme value distribution given by (2.4) at the corresponding standardized stress level ξ_i. Suppose the ordered lifetimes of these n_i units be given by $T_{1:n_i} \leq T_{2:n_i} \leq ... \leq T_{n_i:n_i}$. If we consider Type-I hybrid censoring framework, then the two random variables representing the number of failures and log-censoring time can be denoted by D and $\tau = \min(T_{r:n_i}, t_0)$ respectively, where $t_0 = \ln x_0$ and x_0 is the censoring time. Accordingly, the data can be represented by $(T_{1:n_i}, T_{2:n_i}, ..., T_{D:n_i}, D)$. The likelihood function is expressed as

$$L_i(\mu, \sigma) \propto \prod_{j=1}^{d} f_T(t_{j:n_i})(1 - F_T(\tau_0))^{n-d}, \tag{2.6}$$

where $t_{j:n_i}$, d, and τ_0 are the observed values of $T_{j:n_i}$, D, and τ respectively. Using results from Park and Balakrishnan (2009), the Fisher information matrix at i^{th} stress level is obtained as

$$\ell_i(\boldsymbol{\theta}) = \int_{-\infty}^{t_0} \left(\frac{\partial}{\partial \boldsymbol{\theta}} \ln h_T(t)\right)^{\prime} \left(\frac{\partial}{\partial \boldsymbol{\theta}} \ln h_T(t)\right) \sum_{j=1}^{r} f_{j:n_i}(t) dt; \tag{2.7}$$

where $h_T(t) = \frac{1}{\sigma} e^{\frac{t-\gamma_0-\gamma_1\xi_i}{\sigma}}$ and $f_{j:n_i}(t) = j \binom{n_i}{j} \frac{1}{\sigma} e^{\frac{t-\gamma_0-\gamma_1\xi_i}{\sigma} - (n_i-j+1)e^{\frac{t-\gamma_0-\gamma_1\xi_i}{\sigma}}} \left(1 - e^{-e^{\frac{t-\gamma_0-\gamma_1\xi_i}{\sigma}}}\right)^{j-1}$ are the hazard and density function of T and $T_{j:n_i}$ respectively, and $\boldsymbol{\theta} = (\gamma_0, \gamma_1, \sigma)$. The expression for $\ell(\boldsymbol{\theta})$ is of the form

$$\ell_i(\boldsymbol{\theta}) = \begin{pmatrix} \ell_{11}(\boldsymbol{\theta}) & \ell_{12}(\boldsymbol{\theta}) & \ell_{13}(\boldsymbol{\theta}) \\ \ell_{21}(\boldsymbol{\theta}) & \ell_{22}(\boldsymbol{\theta}) & \ell_{23}(\boldsymbol{\theta}) \\ \ell_{31}(\boldsymbol{\theta}) & \ell_{32}(\boldsymbol{\theta}) & \ell_{33}(\boldsymbol{\theta}) \end{pmatrix};$$

where,

$$\ell_{11}(\boldsymbol{\theta}) = \frac{1}{\sigma^2} \int_{-\infty}^{t_0} \sum_{j=1}^{r} f_{j:n_i}(t)dt,$$

$$\ell_{22}(\boldsymbol{\theta}) = \frac{{\xi_i}^2}{\sigma^2} \int_{-\infty}^{t_0} \sum_{j=1}^{r} f_{j:n_i}(t)dt,$$

$$\ell_{33}(\boldsymbol{\theta}) = \int_{-\infty}^{t_0} \left(\frac{t-\gamma_0-\gamma_1\xi_i}{\sigma^2} + \frac{1}{\sigma}\right)^2 \sum_{j=1}^{r} f_{j:n_i}(t)dt,$$

$$\ell_{12}(\boldsymbol{\theta}) = \ell_{21}(\boldsymbol{\theta}) = \frac{\xi_i}{\sigma^2} \int_{-\infty}^{t_0} \sum_{j=1}^{r} f_{j:n_i}(t)dt,$$

$$\ell_{13}(\boldsymbol{\theta}) = \ell_{31}(\boldsymbol{\theta}) = \frac{1}{\sigma} \int_{-\infty}^{t_0} \left(\frac{t-\gamma_0-\gamma_1\xi_i}{\sigma^2} + \frac{1}{\sigma}\right) \sum_{j=1}^{r} f_{j:n_i}(t)dt,$$

$$\ell_{23}(\boldsymbol{\theta}) = \ell_{32}(\boldsymbol{\theta}) = \frac{\xi_i}{\sigma} \int_{-\infty}^{t_0} \left(\frac{t-\gamma_0-\gamma_1\xi_i}{\sigma^2} + \frac{1}{\sigma}\right) \sum_{j=1}^{r} f_{j:n_i}(t)dt.$$

Let, $n_i = n\pi_i$; where, π_i is the proportion of items to be allocated at i^{th} stress level. Therefore, the Fisher information matrix is obtained as $F = n \sum_{i=1}^{m} \pi_i \ell_i(\boldsymbol{\theta})$. Hence, the variance-covariance matrix can be computed by inverting the Fisher information matrix as

$$F^{-1}(\boldsymbol{\theta}) = \begin{pmatrix} h^{11}(\boldsymbol{\theta}) & h^{12}(\boldsymbol{\theta}) & h^{13}(\boldsymbol{\theta}) \\ h^{21}(\boldsymbol{\theta}) & h^{22}(\boldsymbol{\theta}) & h^{23}(\boldsymbol{\theta}) \\ h^{31}(\boldsymbol{\theta}) & h^{32}(\boldsymbol{\theta}) & h^{33}(\boldsymbol{\theta}) \end{pmatrix}.$$

2.3 Acceptance criterion

Since in case of lifetime as a quality attribute, higher the lifetime of the product, better is its quality. Hence, we only need to be concerned with the lower specification limit (LSL). LSL is the lowest level of product quality that is within the acceptable range. Suppose the actual one-sided LSL be l pertaining to items to be tested, then, the items with lifetimes less than l should be considered nonconforming. Since instead of actual lifetime (X) of the product log-lifetime ($T = \ln X$) is used, therefore, the fraction of nonconforming items, p, can be written as $p = Pr(T \leq l^{'})$, where $l^{'} = \ln l$. Using the lot acceptance criterion derived by Lieberman and Resnikoff (1955) we get the following expression

$$\hat{\mu_0} - k\hat{\sigma} > l^{'}; \tag{2.8}$$

where $\hat{\mu_0}$ and $\hat{\sigma}$ are the maximum likelihood estimates of μ_0 and σ respectively and k is the acceptability constant. The statistic $\hat{S} = \hat{\mu_0} - k\hat{\sigma}$ is asymptotically normal with mean $E[\hat{S}] = \mu_0 - k\sigma$ and variance $Var[\hat{S}] = h^{11}(\hat{\boldsymbol{\theta}}) + k^2 h^{33}(\hat{\boldsymbol{\theta}}) - 2kh^{13}(\hat{\boldsymbol{\theta}})$, where h^{11}, h^{33} and h^{13} are the elements of variance-covariance matrix and $\hat{\boldsymbol{\theta}} = (\hat{\mu_0}, \hat{\sigma})$. So the standardized variate

$$U = \frac{\hat{\mu_0} - k\hat{\sigma} - (\mu_0 - k\sigma)}{\sqrt{h^{11}(\hat{\boldsymbol{\theta}}) + k^2 h^{33}(\hat{\boldsymbol{\theta}}) - 2kh^{13}(\hat{\boldsymbol{\theta}})}} \tag{2.9}$$

is also asymptotically normal with mean 0 and variance 1. Therefore, the expression for OC curve can be represented by

$$\begin{aligned} \mathscr{L}(p) &= Pr(\hat{\mu} - k\hat{\sigma} > l^{'} | p) \\ &= 1 - \Phi\Big(\frac{\sigma(u_p + k)}{\sqrt{V}}\Big); \end{aligned} \tag{2.10}$$

where, $V = h^{11}(\hat{\boldsymbol{\theta}}) + k^2 h^{33}(\hat{\boldsymbol{\theta}}) - 2kh^{13}(\hat{\boldsymbol{\theta}})$ and $u_p = \frac{l^{'} - \mu_0}{\sigma}$ is the p^{th} quantile of the standard extreme value distribution corresponding to the nonconforming fraction $p = Pr((T - \mu_0)/\sigma \leq (l^{'} - \mu_0)/\sigma)$ and $\mathscr{L}(p)$ is decreasing in p and Φ is standard normal distribution function.

If we consider α and β as producer's risk and consumer's risk respectively, then by fixing points $(p_\alpha, 1-\alpha)$ and (p_β, β) on the OC curve we can obtain the value of k.

The expression for k thus obtained can be written as

$$k = \frac{u_{p_\alpha} z_{1-\beta} - u_{p_\beta} z_\alpha}{z_\alpha - z_{1-\beta}}. \tag{2.11}$$

Also, the following equality is obtained in the process.

$$\frac{V}{\sigma^2} \left(\frac{z_\alpha - z_{1-\beta}}{u_{p_\alpha} - u_{p_\beta}} \right)^2 = 1, \tag{2.12}$$

where z_α and $z_{1-\beta}$ are α^{th} and $(1-\beta)^{th}$ quantiles of standard normal distribution and u_{p_α} and u_{p_β} are ${p_\alpha}^{th}$ and ${p_\beta}^{th}$ quantiles of the standard extreme value distribution corresponding to the nonconforming fractions p_α and p_β respectively.

3 ALTSP using cost function approach

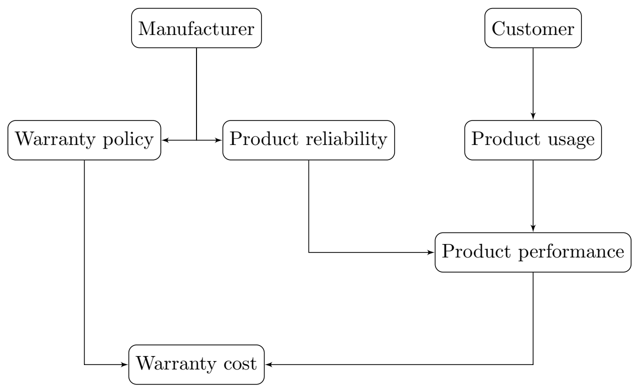
3.1 Problem formulation

Evidence from the literature points out to four primary cost components that are affected by a life test sampling plan. The following are the cost components that constitutes the aggregate cost: $\langle a \rangle$ the acceptance cost, $\langle b \rangle$ rejection cost, $\langle c \rangle$ time-consumption cost, and $\langle d \rangle$ inspection cost. In order to formulate the problem two stress levels are considered, s_1 (to be determined using the proposed model) and $s_2 = s_H$.

If the products are sold under warranty, the decision to accept a lot is going to affect the warranty cost.

The argument can further be strengthened by the following diagram by Murthy (2007).

Figure 2: Characterization of warranty cost



Therefore, warranty cost can be used as a substitute for the cost of acceptance of a lot (Kwon, 1996). The warranty policy using which the warranty cost is derived for this study is called the general rebate warranty. General rebate warranty is a combination of two warranty policies, free-replacement warranty and pro-rata warranty. The mathematical formulation for general rebate warranty is given by the following expression

$$c_a^*(x) = \begin{cases} c_a & x < w_1 \\ c_a \frac{w_2 - x}{w_2 - w_1} & w_1 \leq x < w_2 \\ 0 & x \geq w_2. \end{cases} \tag{3.1}$$

So, if the failure time is less than w_1, the cost incurred for free replacement is c_a. If the product has failure time between interval $[w_1, w_2)$, the cost incurred for pro-rata warranty is in proportion to the difference between failure time and w_2, which is decreasing in nature. If the failure time is beyond w_2, no warranty costs are incurred. Since we use log lifetimes, therefore, according to general rebate warranty policy the cost of accepting an item with log-lifetime t is

$$c_a^*(t) = \begin{cases} c_a & t < \ln w_1 \\ c_a \frac{w_2 - e^t}{w_2 - w_1} & \ln w_1 \leq t < \ln w_2 \\ 0 & t \geq \ln w_2. \end{cases} \tag{3.2}$$

Hence, the expected warranty cost per unit is given by

$$w(\theta) = c_a \Bigg(\frac{w_2 F_T(\ln w_2) - w_1 F_T(\ln w_1)}{w_2 - w_1} - \frac{1}{w_2 - w_1} \int_{\ln w_1}^{\ln w_2} e^t f_T(t) dt \Bigg). \tag{3.3}$$

Thus, the expected warranty (acceptance) cost if n out of N items are put on test is obtained as

$$C_w = (N - n) w(\theta) \Bigg(1 - \Phi\Big(\frac{\sigma(u_p + k)}{\sqrt{V}} \Big) \Bigg). \tag{3.4}$$

From the literature, rejection cost usually is taken as cost due to units that are not tested (Hsieh and Lu, 2013). Thus, if c_r is the cost per unit for the items that are not put on test, then the average cost of rejecting a lot is given by

$$C_r = (N-n)c_r\Phi\Big(\frac{\sigma(u_p+k)}{\sqrt{V}}\Big). \tag{3.5}$$

The expected log-time of the test is given by

$$\begin{aligned} E[\tau] &= [min(T_{r:n}, t_0)] \\ &= t_0 P(T_{r:n} \geq t_0) + E[T_{r:n}|T_{r:n} < t_0]P(T_{r:n} < t_0) \\ &= t_0\Big(1 - \sum_{j=r}^{n}\binom{n}{j}F_T(t_0)^j\big(1-F_T(t_0)\big)^{n-j}\Big) + r\binom{n}{r}\int_0^{t_0} tF_T(t)^{r-1}\big(1-F_T(t)\big)^{n-r}f_T(t)dt. \end{aligned} \tag{3.6}$$

Now, if c_t be the cost per unit, the expression for expected time consumption cost is given by $C_t = c_t E[\tau]$. Also, if c_i is the unit cost of inspection, the average cost of inspection can be written as $C_i = nc_i$. Therefore, the aggregate cost function is

$$\begin{aligned} TC(n, r, t_0, \xi_1, \pi_1) &= C_w + C_r + C_t + C_i \\ &= (N-n)w(\theta)\left(1 - \Phi\Big(\frac{\sigma(u_p+k)}{\sqrt{V}}\Big)\right) + (N-n)c_r\Phi\Big(\frac{\sigma(u_p+k)}{\sqrt{V}}\Big) + c_t E[\tau] + nc_i \\ &= (N-n)\left(w(\theta) + \big(c_r - w(\theta)\big)\Phi\Big(\frac{\sigma(u_p+k)}{\sqrt{V}}\Big)\right) + c_t E[\tau] + nc_i \end{aligned}$$

Hence, the optimal design problem can be expressed as follows:

$$\textit{minimize } TC(n, r, t_0, \xi_1, \pi_1)$$

$$\textit{subject to } \frac{V}{\sigma^2}\left(\frac{z_\alpha - z_{1-\beta}}{u_{p_\alpha} - u_{p_\beta}}\right)^2 - 1 = 0.$$

The equality constraint as also shown in (2.12) ensures that the already agreed upon values pertaining to producer's and consumer's risks are being maintained.

3.2 Determining the optimal solution

The optimization problem described in the aforementioned subsection is mixed-integer non-linear in nature. This dual nature of the problem enhances its complexity. Therefore, in order to reduce its complexity, instead of using n as a decision variable $p_n = \frac{n}{N}$ is used. To retain the integer nature of n, n is replaced with $\lfloor p_n^* N \rfloor$, where $\lfloor . \rfloor$ represents greatest integer or the floor function. Similarly, instead of r as a decision variable, the degree of censoring, $q = 1 - \frac{r}{n}$ is used and r is replaced with $\lfloor (1-q)n \rfloor$ to retain its discrete nature. The continuous nature of p_n ($p_n \in [0, 1]$) and q ($q \in [0, 1]$) transforms the problem to a nonlinear programming problem. Hence, the problem gets transformed to

$$\textit{minimize } TC(p_n, q, t_0, \xi_1, \pi_1)$$

$$\textit{subject to } \frac{V}{\sigma^2}\left(\frac{z_\alpha - z_{1-\beta}}{u_{p_\alpha} - u_{p_\beta}}\right)^2 - 1 = 0.$$

The solution procedure is summarized using the following algorithm.

Algorithm 1: Finding the optimal design using cost function approach.

Input: p, α, β, p_α, and p_β, N.

Output: n^*, r^*, ${x_0}^*$, ${\xi_1}^*$, ${\pi_1}^*$, and TC^*

1 Define functions f_T, F_T, and $\mathscr{L}(p)$

2 Fix (α, p_α) and (β, p_β) in $\mathscr{L}(p)$ to find k and and the constraint function

3 Set w_1, w_2 and unit costs

4 Consider $p_n = \frac{n}{N}$ and $q = 1 - \frac{r}{n}$ as decision variables

5 Replace n with $\lfloor p_n N \rfloor$ and r with $\lfloor (1-q)n \rfloor$ to transform the objective function from $TC(n, r, t_0, \xi_1, \pi_1)$ to $TC(p_n, q, t_0, \xi_1, \pi_1)$

6 Minimize the objective function with respect to the given constraint to find the optimal values of (${p_n}^*$, q^*, ${t_0}^*$, ${\xi_1}^*$,${\pi_1}^*$, TC^*) using non-linear optimization algorithms such as augmented Lagrangian

7 Obtain $n^* = \lfloor p_n^* N \rfloor$, $r^* = \lfloor (1-q)n^* \rfloor$ and ${x_0}^* = e^{{t_0}^*}$ to find the optimal design $(n^*, r^*, {x_0}^*, {\xi_1}^*, {\pi_1}^*)$

The values of p_α and p_β are usually decided jointly by the producer and the consumer. But for the purpose of our study we have used the values from MIL-STD-105D (U D of Defense, 1963). The optimal values found using the aforementioned approach is summarized in Table 1.

Table 1: ALTSP using cost function approach for given values of α, β, p_α, and p_β

(α, β)	(p_α, p_β)	p_n^*	n^*	q^*	r^*	${\xi_1}^*$	${\pi_1}^*$	n_1	n_2	${t_0}^*$	${x_0}^*$	TC^*
	(0.0209, 0.0742)	0.103	52	0.109	46	0.545	0.278	15	37	3.283	26.667	28.568
$(0.05, 0.1)$	(0.0319, 0.0942)	0.112	56	0.140	48	0.676	0.393	22	34	3.292	26.917	28.532
	(0.0190, 0.0535)	0.101	50	0.111	45	0.642	0.415	20	30	3.311	27.421	28.577
	(0.0209, 0.0742)	0.126	63	0.128	54	0.511	0.326	20	43	3.003	25.145	45.538
$(0.1, 0.1)$	(0.0319, 0.0942)	0.106	53	0.133	46	0.464	0.366	19	34	3.288	26.792	46.032
	(0.0190, 0.0535)	0.108	54	0.114	47	0.638	0.485	26	28	3.182	24.093	45.977

The two most extensively used values for (α, β) are chosen from the literature to obtain the ALTSPs. For numerical illustration, the result summarized in the first row of Table 1 is explained as follows: Consider a lot of size $N = 500$. A decision on whether the lot is fit to be accepted has to be made using accelerated life test with two stress levels. In order to arrive at the decision under the given setting, 52 (n) items are to be tested. Out of 52 testing units, 15 (n_1) are to be tested at the standardized stress level of 0.545 (${\xi_1}^*$) and the rest of the items ($n_2 = 37$) are to be tested at the highest stress level. The experiment is terminated either when 46 (r) items fail or when 3.283 (t_0) units of time has elapsed.

The solutions summarized in Table 1 are obtained using *nloptr* package in *R*. The *nloptr* package solves non-linear optimization problems with linear and/or non-linear constraints. The *auglag* function within *nloptr* package uses augmented Lagrangian minimization for optimizing nonlinear objective functions with constraints. This method modifies the given objective function by combining the constraint function to it. The modified objective function is then fed to another optimization algorithm. The three most commonly used algorithms are COBYLA (Constrained optimization by linear approximation), LBFGS (Limited-memory (BFGS)) and MMA (Method of moving asymptotes). Detailed understanding of the usage of these algorithms is discussed in Ypma (2014). An assessment of the performance of these algorithms in terms of computation time is noted in Table 2.

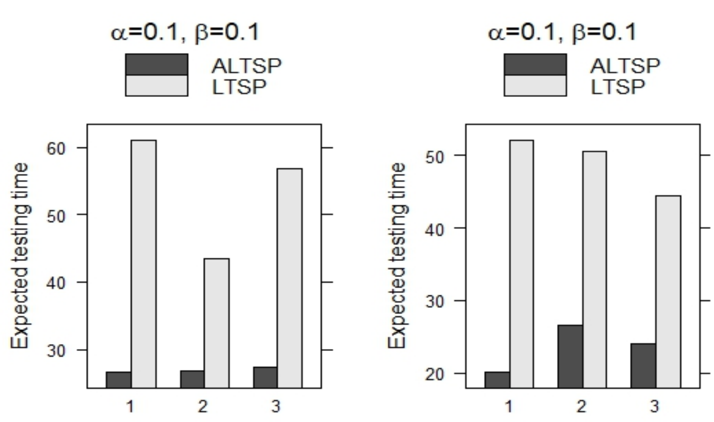
Table 2: Computation times for COBYLA, LBFGS, and MMA in seconds

(α, β)	(p_α, p_β)	COBYLA	LBFGS	MMA
	(0.0209, 0.0742)	265.54	700.82	640.97
$(0.05, 0.1)$	(0.0319, 0.0942)	296.69	846.95	562.31
	(0.0190, 0.0535)	260.06	856.64	648.63
	(0.0209, 0.0742)	298.39	808.79	590.57
$(0.1, 0.1)$	(0.0319, 0.0942)	291.28	886.09	562.16
	(0.0190, 0.0535)	277.57	871.06	593.10

From Table 2 it is observed that for the given problem, COBYLA performs better in terms of total clock time. All the subsequent problems discussed in this paper are solved using *nloptr* package in *R*.

The main purpose of using accelerated life tests in order to develop life test sampling plans is to reduce the duration of the tests. In order to ensure that the intended purpose is served, the values of expected testing times are obtained under the given setup for both accelerated and usage stress settings. The results outlined in Figure 3 are as desired.

Figure 3: Cost comparison between the two approaches



When $r = n$, Type-I hybrid censoring scheme transforms to Type-I censoring scheme. Hence as a special case, the optimum ALTSPs for Type-I censoring scheme is obtained using the given setup. The results attained are summarized in Table 3.

Table 3: ALTSP using cost function approach for given values of α, β, p_α, and p_β

(α,β)	(p_α,p_β)	p_n^*	n^*	${\xi_1}^*$	${\pi_1}^*$	n_1	n_2	${t_0}^*$	${x_0}^*$	TC^*
	(0.0209, 0.0742)	0.102	51	0.587	0.489	25	26	3.227	25.225	28.563
$(0.05, 0.1)$	(0.0319, 0.0942)	0.102	51	0.592	0.438	22	29	3.342	28.278	28.572
	(0.0190, 0.0535)	0.101	50	0.675	0.397	20	30	3.320	27.671	28.578
	(0.0209, 0.0742)	0.116	58	0.613	0.468	27	31	3.096	22.115	45.769
$(0.1, 0.1)$	(0.0319, 0.0942)	0.114	56	0.657	0.388	22	34	3.277	26.499	45.859
	(0.0190, 0.0535)	0.106	53	0.648	0.457	24	29	3.199	24.517	46.010

4 ALTSP using variance minimization approach

4.1 Variance minimization without cost constraint

As an alternative approach, modified variance measure proposed by Kundu (2008) is minimized to obtain the ALTSPs. The measure was originally introduced by Zhang and Meeker (2005) as

$$Var[\ln \hat{X}_m],$$

where $\hat{X}_m$ is the maximum likelihood estimate of the m^{th} quantile of the lifetime distribution. Kundu (2008) modified this measure to

$$\int_0^1 Var[\ln \hat{X}_m]dm,$$

where the integral over $[0,1]$ represents an aggregate variance of the quantile estimates over all quantile points. It is important to note that the modified variance measure is not dependent on m but on the decision variables. $\ln \hat{X}_m$ can be expressed as $\ln \hat{X}_m = \hat{\mu} + \hat{\sigma} g_1(m)$, where $g_1(m) = \ln[-\ln(1-m)]$. Therefore,

$$\int_0^1 Var[\ln \hat{X}_m]dm = h^{11}(\hat{\boldsymbol{\theta}}) + h^{33}(\hat{\boldsymbol{\theta}}) \int_0^1 \big(g_1(m)\big)^2 dm + 2h^{13}(\hat{\boldsymbol{\theta}}) \int_0^1 \big(g_1(m)\big) dm. \tag{4.1}$$

This measure has been used in the literature to obtain life test sampling plans under various censoring schemes. In the context of Type-I hybrid censoring scheme, Bhattacharya *et al* (2014) and Bhattacharya *et al* (2015) used this measure to develop life test sampling plans. Using this measure, the ALTSP problem under the given setting is obtained as

$$minimize \int_0^1 Var[\ln \hat{X}_m]dm$$

$$subject\ to\ \frac{V}{\sigma^2}\left(\frac{z_\alpha - z_{1-\beta}}{u_{p_\alpha} - u_{p_\beta}}\right)^2 - 1 = 0.$$

The solution to the problem, summarized in Table 4, is obtained using a similar approach used in the previous section. The steps required to obtain the optimal design is mentioned in Algorithm 2.

Table 4: ALTSP using variance minimization approach for given values of α, β, p_α, and p_β

(α, β)	(p_α, p_β)	p_n^*	n^*	q^*	r^*	${\xi_1}^*$	${\pi_1}^*$	n_1	n_2	${t_0}^*$	${x_0}^*$	Var^*
$(0.05, 0.1)$	(0.0209, 0.0742)	0.202	101	0.108	90	0.683	0.471	47	54	2.887	17.950	0.137
	(0.0319, 0.0942)	0.203	101	0.103	91	0.472	0.430	43	58	2.998	20.055	0.091
	(0.0190, 0.0535)	0.200	100	0.103	89	0.689	0.431	43	57	2.947	19.042	0.113
$(0.1, 0.1)$	(0.0209, 0.0742)	0.199	99	0.105	89	0.690	0.371	37	62	2.911	18.382	0.127
	(0.0319, 0.0942)	0.204	102	0.103	91	0.689	0.404	41	61	2.921	18.564	0.116
	(0.0190, 0.0535)	0.214	107	0.104	95	0.694	0.468	49	58	2.843	17.162	0.117

Algorithm 2: Finding the optimal design using variance minimization approach.

Input: p, α, β, p_α, and p_β, N.

Output: n^*, r^*, ${x_0}^*$, ${\xi_1}^*$, ${\pi_1}^*$, and Var^*

1 Define functions f_T, F_T, and $\mathscr{L}(p)$

2 Fix $(\alpha,\ p_\alpha)$ and $(\beta,\ p_\beta)$ in $\mathscr{L}(p)$ to find k and and the constraint function

3 Set w_1, w_2 and unit costs

4 Consider $p_n = \frac{n}{N}$ and $q = 1 - \frac{r}{n}$ as decision variables

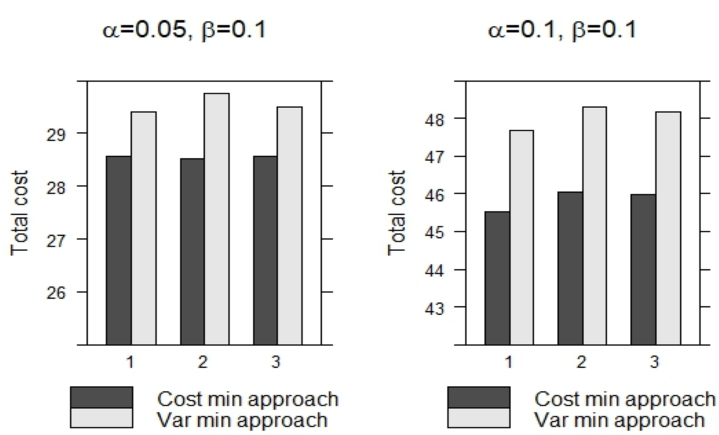
5 Replace n with $\lfloor p_n N \rfloor$ and r with $\lfloor (1-q)n \rfloor$ to transform the problem from $\underset{n,r,t_0,\xi_1,\pi_1}{minimize} \int_0^1 Var[\ln \hat{X}_m]dm$ to $\underset{p_n,q,t_0,\xi_1,\pi_1}{minimize} \int_0^1 Var[\ln \hat{X}_m]dm$

6 Minimize the objective function with respect to the given constraint to find the optimal values of $({p_n}^*,\ q^*,\ {t_0}^*,\ {\xi_1}^*, {\pi_1}^*,\ Var^*)$ using non-linear optimization algorithms such as augmented Lagrangian

7 Obtain $n^* = \lfloor p_n^* N \rfloor$, $r^* = \lfloor (1-q)n^* \rfloor$ and ${x_0}^* = e^{{t_0}^*}$ to find the optimal design $(n^*, r^*, {x_0}^*, {\xi_1}^*, {\pi_1}^*)$

It is evident from the tables (Table 1 and Table 3) that the optimum number of items to be tested is more when obtained using variance minimization approach, whereas the optimum censoring time is less when obtained using the same.

Figure 4: Comparison of expected testing time



On comparison of the total cost between the two approaches, it is found that the total cost is higher in case of variance minimization approach. The cost comparison is depicted using Figure 4.

4.2 Variance minimization without cost constraint

It is rationale for a decision maker to use cost as a constraint to negate the possibility of incurring exceedingly high cost while using variance minimization approach. Therefore, in order to keep a check on the total cost, the minimization problem discussed the previous subsection is reintroduced by including the total cost as a constraint. Hence, the problem can be expressed as

$$\mathit{minimize} \int_0^1 Var[\ln \hat{X}_m] dm$$

$$\mathit{subject\ to}\ \frac{V}{\sigma^2}\left(\frac{z_\alpha - z_{1-\beta}}{u_{p_\alpha} - u_{p_\beta}}\right)^2 - 1 = 0,$$

$$TC(n, r, t_0, \xi_1, \pi_1) \leq C_0.$$

The solution to the problem is obtained using Algorithm 2. The optimal designs determined for different values of C_0 are summarized in Table 4.

Table 5: ALTSP using variance minimization approach with cost constraint for given values of α, β, p_α, and p_β

(p_α, p_β)	C_0	p_n^*	n^*	q^*	r^*	${\xi_1}^*$	${\pi_1}^*$	n_1	n_2	${t_0}^*$	${x_0}^*$	Var^*
(α,β)=(0.05, 0.1)												
(0.0209, 0.0742)	20	0.199	99	0.104	89	0.316	0.417	40	59	2.966	19.413	0.148
	25	0.204	102	0.123	89	0.457	0.468	47	55	2.898	18.145	0.145
	30	0.208	104	0.105	93	0.685	0.476	49	55	2.873	17.701	0.136
(0.0319, 0.0942)	20	0.202	100	0.103	90	0.299	0.398	39	61	3.101	22.229	0.113
	25	0.200	100	0.107	89	0.394	0.418	41	59	3.044	20.998	0.108
	30	0.203	101	0.105	90	0.470	0.421	42	59	2.997	20.043	0.091
(0.0190, 0.0535)	20	0.201	100	0.101	90	0.302	0.411	40	60	3.001	20.101	0.130
	25	0.198	99	0.104	89	0.475	0.415	40	59	2.964	19.377	0.125
	30	0.201	100	0.105	89	0.683	0.431	43	56	2.948	19.080	0.113
(α,β)=(0.1, 0.1)												
(p_α, p_β)	C_0	p_n^*	n^*	q^*	r^*	${\xi_1}^*$	${\pi_1}^*$	n_1	n_2	${t_0}^*$	${x_0}^*$	Var^*
(0.0209, 0.0742)	40	0.197	98	0.101	88	0.302	0.396	39	59	2.995	19.996	0.142
	45	0.198	99	0.102	89	0.475	0.417	40	59	2.949	19.087	0.136
	50	0.198	99	0.104	88	0.690	0.414	40	59	2.901	18.182	0.128
(0.0319, 0.0942)	40	0.201	100	0.101	90	0.303	0.403	40	60	2.945	19.013	0.131
	45	0.201	100	0.111	89	0.457	0.398	40	60	2.946	19.025	0.123
	50	0.204	102	0.104	91	0.698	0.412	41	61	2.933	18.785	0.116
(0.0190, 0.0535)	40	0.200	100	0.102	89	0.324	0.410	41	59	2.894	18.072	0.138
	45	0.197	98	0.101	88	0.441	0.396	39	59	2.995	19.987	0.130
	50	0.200	100	0.101	90	0.695	0.448	45	54	2.834	17.026	0.118

From Table 4 it is observed that as the cost constraint (C_0) is increased, the optimum stress level (${\xi_1}^*$) increases and the optimum variance measure (Var^*) decreases.

5 Sensitivity analysis and other numerical observations

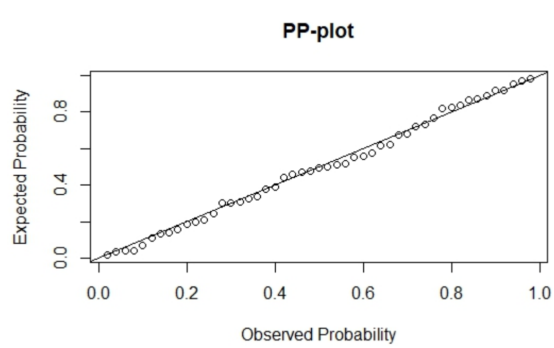
In this section, sensitivity analysis is performed for the cost minimization approach proposed for finding ALTSPs for products under warranty. Further, the impact of inclusion of warranty cost on optimal design is also observed.

While developing ALTSPs, the parameters of the extreme value distribution are to be relied upon. Hence, to investigate the effect of mis-specification of parameters in the optimum design and the total cost, we incorporate a sensitivity analysis study. For the purpose of this study, a real life failure dataset pertaining to low-metallic break pads of a car model is used. Each point of the dataset represents the number of thousand kilometers traversed before a brake pad fails in a life testing experiment. The data depicting the failure times (in thousand of kilometers) of the units failed during the life test experiment is represented as follows:

53.342, 60.122, 54.139, 68.564, 58.448, 61.068, 60.945, 66.850, 53.975, 50.355, 57.675, 37.295, 58.704, 64.134, 59.774, 48.541, 51.080, 44.137, 41.613, 58.459, 50.849, 66.997, 53.925, 63.419, 41.425, 65.482, 72.973, 49.233, 54.527, 52.263, 57.315, 55.766, 62.065, 48.101, 59.625, 71.441, 54.780, 63.143, 66.001, 65.651, 40.370, 57.995, 58.824, 70.545, 68.625, 47.081, 67.548, 62.251, 57.910, and 56.144.

Given the dataset, we estimate the parameter values $(\hat{\mu}_0, \hat{\sigma}) = (4.114, 0.124)$ assuming the life-time to follow Weibull distribution. In order to ensure that the distributional assumption (to arrive at the estimates) holds true, we draw a PP plot. The PP plot depicted in Figure 5 shows a good fit and hence validates that the distributional assumption.

Figure 5: PP plot of the dataset using Weibull distribution



To further ensure that the assumption holds true, the Kolmogorov-Smirnov (KS) distance statistic value between the empirical distribution function and the fitted distribution function is obtained. The value of KS distance is found to be 0.056 with the associated p-value as 0.9952. ALTSPs are obtained for three sets of (μ_0, σ) values (estimated parameter values, 1.2 $\times$ estimated parameter values, and 1.4 $\times$ estimated parameter values) each at two levels of γ_1 (0.5 and 0.75). Although the optimal design changes with change in parameter values, no significant trend or pattern is visible. But a clear trend emerges for the values of optimal cost which is depicted using Figure 6.

Figure 6: Change in optimal cost due to change in parameters

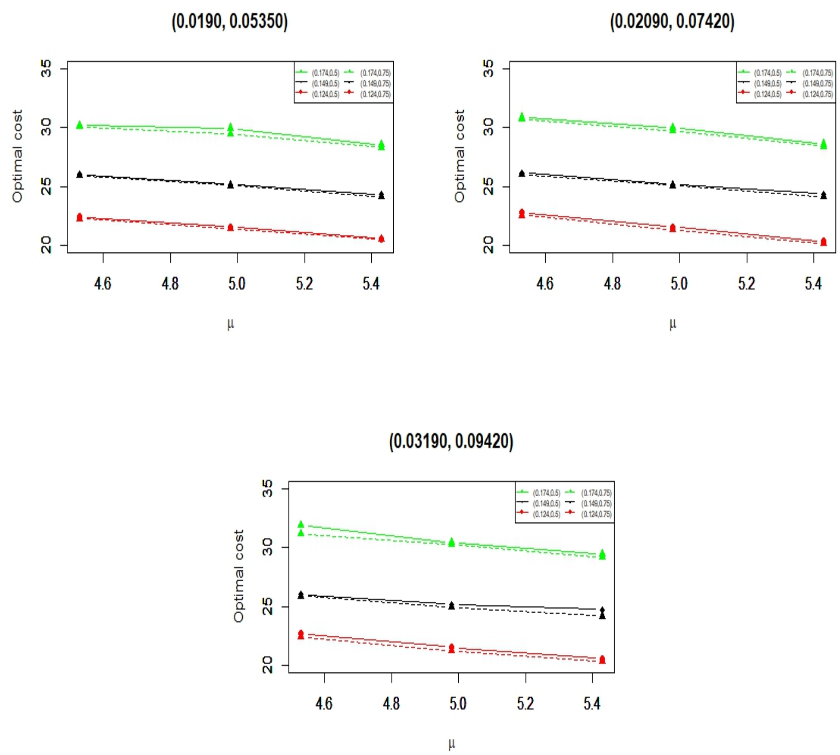


Figure 6 shows that for each of the three sets of (p_α, p_β) values pertaining to $(\alpha = 0.05, \beta = 0.1)$, the following trends are visible.

- The optimal cost decreases when the values of μ_0 increases keeping the other two parameter fixed. This trend is evident from the downward sloping lines in the figure.

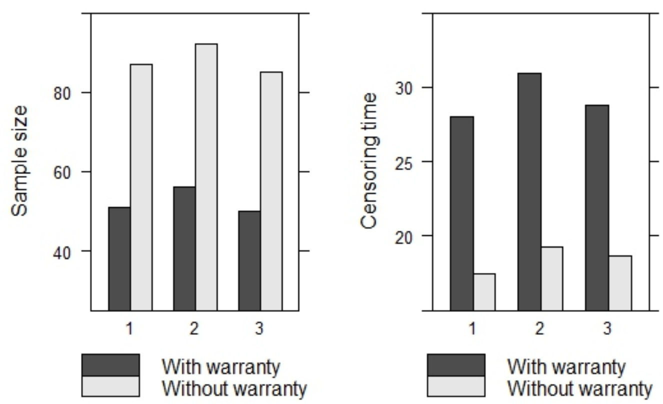
- The optimal cost increases with the increase in the value of σ when the other parameters are kept constant. The parallel lines in the figure describes this phenomenon.

- The optimal cost slightly decreases with the increase in γ_1 when the other parameters are kept unchanged. The dotted lines in the figure resembles the optimal cost obtained at a higher value of γ_1 which are slightly lower than the respective smooth lines which are at a lower value of γ_1.

The results obtained for the aforementioned analysis are summarized in Table 7 in the Appendix.

An important aspect of the proposed approach is the inclusion of warranty cost in the formulation of the problem. Therefore, the change in optimal design due to inclusion of warranty cost is measured. Figure 7 illustrates the change in optimal sample size and optimal censoring time when the optimal design is obtained with and without including warranty cost. It is found that exclusion of warranty cost, the optimal sample size to be tested increases whereas the optimal censoring time decreases.

Figure 7: Change in optimal design due to warranty cost



6 Case illustration

In order to explain the practical use of the model a real life situation is presented using the following case.

A car company in India decided to switch from high-metallic brake-pads to low-metallic brake-pads. The decision to make this switch resulted from repeated complaints from the customers regarding noise, vibration and harshness resulting from metal to metal contact of which also affected the life of high-metallic brake-pads. The low-metallic brake-pads are expected to improve customer satisfaction by providing better life and barking performance. The car company ordered the low-metallic brake-pads from one of its suppliers and also indicated that upon validation of the pad-life all its car models will be using low-metallic brake-pads. The car company is willing to conduct an accelerated life test at two stress levels in order to ensure that the intended quality level in terms of lifetime of the pads is maintained since the brake-pads are under warranty.

The following questions are to be answered before conducting the accelerated life test.

- How many items are to be tested?

- At what time the experiment is to be terminated?
- What should be the optimum stress level?
- How many items are to be tested at each stress levels?

Intending to answer the above questions using the proposed model, the following assumptions are made:

- The parameter values ($\mu_0 = 4.11, \sigma = 0.12, \gamma_1 = 0.50$) are known to the company.
- The values of (α, β) and (p_α, p_β) are fixed at (0.05, 0.1) and (0.0209, 0.0742) respectively.

The proposed model is used to get the results mentioned in Table 6.

Table 6: Results obtained for the aforementioned problem using cost function approach

p_n^*	n^*	q^*	r^*	${\xi_1}^*$	${\pi_1}^*$	n_1	n_2	${t_0}^*$	${x_0}^*$	TC^*
0.103	51	0.118	45	0.654	0.483	24	27	3.332	28.001	30.84

Using results from Table 6, it is obtained that 51 units are required to be tested. Of the 51 units, 24 units are to be tested at a stress level of 0.654 and 27 units at the highest stress level. The test is terminated either when 45 items fail or when 28 units of time elapses.

7 Conclusion

In this study a method is proposed to arrive at an optimum ALTSP Under Type-I hybrid censoring. Weibull lifetime model is considered in the context of this study, however under the ambit of the developed methodology other lifetime distributions of log-location scale family can also be used. The work tries to formulate optimum reliability acceptance sampling plans from a management perspective which makes it valuable in dealing with real life problems pertaining to product quality management. One such real life situation is illustrated in Section 6. To get further insights from the model a rigorous sensitivity study is conducted using a real life dataset. The results from the study highlights the importance of the parameters and accesses the behavior of the optimal cost due to parameter changes. Insights on the behavior of optimal cost due to change in period of warranty is also highlighted. A significant change in optimal design is observed after inclusion of warranty cost in the model.

Warranty claims lead to rework, as a result of which cost in terms of efforts, time and money has to be borne by the company. Hence, for consumer durable products it is important to ensure that the cost due warranty is induced in designing the life testing plan. Therefore, from quality management perspective this study takes a small step forward in the direction of addressing a practical problem.

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Appendix

Table 7: Results of sensitivity analysis using brake-pad failure data

$(\mu_0, \sigma, \gamma_1)$	${p_n}^*$	n^*	q^*	r^*	ξ^*	π^*	n_0	n_1	t_0^*	x_0^*	TC^*
(p_α, p_β)=(0.0190, 0.05350)											
(4.114,0.124,0.50)	0.117	58	0.105	52	0.657	0.461	27	31	3.360	28.802	30.169
(4.114,0.149,0.50)	0.125	62	0.131	54	0.673	0.492	30	31	3.289	26.831	26.019
(4.114,0.174,0.50)	0.112	56	0.117	49	0.639	0.363	20	35	3.181	24.064	22.403
(4.937,0.124,0.50)	0.121	60	0.144	51	0.685	0.316	18	42	4.019	55.679	29.940
(4.937,0.149,0.50)	0.119	59	0.191	48	0.640	0.441	26	33	3.932	50.995	25.159
(4.937,0.174,0.50)	0.123	61	0.109	54	0.567	0.441	27	34	3.876	48.236	21.562
(5.761,0.124,0.50)	0.120	60	0.183	49	0.574	0.419	25	34	4.518	91.665	28.527
(5.761,0.149,0.50)	0.121	60	0.163	50	0.548	0.458	27	33	4.353	77.785	24.322
(5.761,0.174,0.50)	0.113	56	0.125	49	0.663	0.422	24	32	4.308	74.350	20.547
(4.114,0.124,0.75)	0.102	51	0.109	45	0.495	0.443	22	29	3.351	28.527	30.058
(4.114,0.149,0.75)	0.108	54	0.136	46	0.649	0.459	25	29	3.257	25.982	25.918
(4.114,0.174,0.75)	0.106	53	0.110	47	0.648	0.469	25	28	3.213	24.876	22.279
(4.937,0.124,0.75)	0.101	50	0.119	44	0.454	0.383	18	32	4.027	56.121	29.446
(4.937,0.149,0.75)	0.101	50	0.127	44	0.614	0.389	19	31	3.941	51.480	25.057
(4.937,0.174,0.75)	0.106	53	0.113	47	0.598	0.336	17	36	3.905	49.674	21.415
(5.761,0.124,0.75)	0.103	51	0.196	41	0.606	0.439	22	29	4.512	91.192	28.310
(5.761,0.149,0.75)	0.109	54	0.198	44	0.684	0.415	22	32	4.413	82.583	24.123
(5.761,0.174,0.75)	0.107	53	0.101	48	0.639	0.341	18	35	4.354	77.839	20.485
(p_α, p_β)=(0.02090, 0.07420)											
(4.114,0.124,0.50)	0.103	51	0.118	45	0.654	0.483	24	27	3.332	28.001	30.840
(4.114,0.149,0.50)	0.110	55	0.116	48	0.626	0.393	21	33	3.197	24.459	26.180
(4.114,0.174,0.50)	0.106	53	0.102	47	0.664	0.452	24	29	3.164	23.676	22.771
(4.937,0.124,0.50)	0.101	50	0.100	45	0.681	0.350	17	33	4.001	54.670	29.959
(4.937,0.149,0.50)	0.108	54	0.100	48	0.682	0.415	22	32	3.898	49.329	25.147
(4.937,0.174,0.50)	0.102	51	0.128	44	0.679	0.440	22	28	3.824	45.811	21.572
(5.761,0.124,0.50)	0.103	51	0.124	45	0.690	0.398	20	31	4.452	85.796	28.601
(5.761,0.149,0.50)	0.107	53	0.189	43	0.626	0.316	17	36	4.338	76.579	24.349
(5.761,0.174,0.50)	0.107	53	0.138	46	0.665	0.325	17	36	4.243	69.627	20.311
(4.114,0.124,0.75)	0.106	53	0.115	47	0.661	0.495	26	27	3.332	28.005	30.689
(4.114,0.149,0.75)	0.105	52	0.133	45	0.657	0.484	25	27	3.238	25.505	26.013
(4.114,0.174,0.75)	0.112	56	0.120	49	0.567	0.461	26	30	3.106	22.326	22.536
(4.937,0.124,0.75)	0.103	50	0.102	45	0.687	0.498	25	25	4.025	56.014	29.734
(4.937,0.149,0.75)	0.101	50	0.101	45	0.598	0.389	19	30	3.914	50.108	25.077
(4.937,0.174,0.75)	0.105	52	0.242	40	0.682	0.424	21	31	3.815	45.378	21.301
(5.761,0.124,0.75)	0.101	50	0.220	39	0.599	0.447	22	28	4.485	88.737	28.420
(5.761,0.149,0.75)	0.104	51	0.101	46	0.525	0.452	23	28	4.389	80.595	24.121
(5.761,0.174,0.75)	0.100	50	0.104	44	0.509	0.355	17	32	4.265	71.210	20.125
(p_α, p_β)=(0.03190, 0.09420)											
(4.114,0.124,0.50)	0.108	54	0.102	48	0.665	0.389	21	33	3.432	30.962	31.894
(4.114,0.149,0.50)	0.109	54	0.104	48	0.674	0.346	18	36	3.273	26.404	26.013
(4.114,0.174,0.50)	0.111	55	0.131	48	0.673	0.483	26	29	3.172	23.855	22.711
(4.937,0.124,0.50)	0.102	50	0.117	44	0.604	0.417	20	30	4.031	56.344	30.439
(4.937,0.149,0.50)	0.102	50	0.113	45	0.643	0.445	22	28	3.994	54.318	25.165
(4.937,0.174,0.50)	0.108	54	0.106	48	0.654	0.476	25	28	3.878	48.339	21.520
(5.761,0.124,0.50)	0.106	52	0.122	46	0.611	0.444	23	29	4.544	94.113	29.465
(5.761,0.149,0.50)	0.102	50	0.232	39	0.679	0.471	24	26	4.461	86.598	24.729
(5.761,0.174,0.50)	0.105	52	0.101	47	0.692	0.424	21	31	4.342	76.906	20.540
(4.114,0.124,0.75)	0.104	52	0.147	44	0.573	0.453	23	28	3.422	30.642	31.151
(4.114,0.149,0.75)	0.108	54	0.112	47	0.582	0.421	22	32	3.331	27.981	25.890
(4.114,0.174,0.75)	0.105	52	0.112	46	0.479	0.446	23	29	3.202	24.594	22.379
(4.937,0.124,0.75)	0.102	51	0.163	42	0.693	0.496	25	26	4.041	56.872	30.271
(4.937,0.149,0.75)	0.102	51	0.255	38	0.637	0.492	25	26	4.002	54.742	24.929
(4.937,0.174,0.75)	0.109	54	0.148	46	0.499	0.484	26	28	3.868	47.885	21.234
(5.761,0.124,0.75)	0.101	50	0.100	45	0.613	0.349	18	32	4.520	91.878	29.153
(5.761,0.149,0.75)	0.103	51	0.108	46	0.663	0.388	20	31	4.477	88.031	24.168
(5.761,0.174,0.75)	0.100	50	0.126	44	0.567	0.475	23	26	4.377	79.602	20.352

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