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Stochastic Comparison of Parallel Systems with Finite Range Distributed Components

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Abstract

In this paper, we study stochastic comparisons of parallel systems having log-Lindley distributed components. These comparisons are carried out with respect to reversed hazard rate and likelihood ratio ordering.

1 Introduction

In reliability optimization and life testing experiments, many times the tests are censored or truncated when failure of a device during the warranty period may not be counted or items may be replaced after a certain time under a replacement policy. Moreover, many reliability systems and biological organism including human life span are bounded above because of test conditions, cost or other constraints. These situations result in a data set which is modeled by distributions with finite range (i.e. with bounded support) viz. power function density, finite range density, truncated Weibull, beta, Kumaraswamy and so on (see for example, Ghitany [5], Lai and Jones [10], Lai and Mukherjee [11], Moore and Lai [17] and Mukherjee and Islam [18]).

Recently, Gómez et al. [6] introduce the log-Lindley (LL) distribution with parameters (σ, λ), written as $\mathrm{LL}(\sigma, \lambda)$, as an alternative to the beta distribution with the probability density function given by

$$f(x; \sigma, \lambda) = \frac{\sigma^2}{1 + \lambda\sigma} \left(\lambda - \log x\right) x^{\sigma-1}; \ 0 < x < 1, \ \lambda \geq 0, \ \sigma > 0, \tag{1.1}$$

where σ is the shape parameter and λ is the scale parameter. This distribution with a simple expression and nice reliability properties, is derived from the generalized Lindley distribution as proposed by Zakerzadeh and Dolati [21], which is again a generalization of the Lindley distribution as proposed by Lindley [14]. The LL distribution exhibits bath-tub failure rates and has increasing generalized failure rate (IGFR). This distribution has useful applications in the context of inventory management, pricing and supply chain contracting problems (see, for example, Ziya et al. [22], Lariviere and Porteus [12] and Lariviere [13]), where a demand distribution is required to have the IGFR property. Moreover, it has application in the actuarial context where the cumulative distribution function (CDF) of the LL distribution is used to distort the premium principle (Gómez et al. [6]). The LL distribution is also shown to fit rates and proportions data better than the beta distribution (Gómez et al. [6]).

Order statistics play an important role in reliability optimization, life testing, operations research and many other areas. Parallel and series systems are the building blocks of many complex coherent systems in reliability theory. While the lifetime of a series system corresponds to the smallest order statistic $X_{1:n}$, the same of a parallel system is represented by the largest order statistic $X_{n:n}$. Although stochastic comparisons of order statistics from homogeneous populations have been studied in detail in the literature, not much work is available so far for the same from heterogeneous populations, due to its complicated nature of expressions. Such comparisons are studied with exponential, gamma, Weibull, generalized exponential or Fréchet distributed components with unbounded support. One may refer to Dykstra *et al.* [2], Misra and Misra [16], Zhao and Balakrishnan ([23]), Torrado and Kochar [20], Kundu and Chowdhury [8], Kundu *et al.* [9], Gupta *et al.* [7] and the references there in. Moreover, not much attention has been paid so far to the stochastic comparison of two systems having finite range distributed components. The notion of majorization (Marshall et al. [5]) is also essential

to the understanding of the stochastic inequalities for comparing order statistics. This concept is used in the context of optimal component allocation in parallel-series as well as in series-parallel systems, allocation of standby in series and parallel systems, and so on, see, for instance, El-Neweihi et al. [3]. It is also used in the context of minimal repair of two-component parallel system with exponentially distributed lifetime by Boland and El-Neweihi [1].

In this paper our main aim is to compare two parallel systems in terms of reversed hazard rate order and likelihood ratio order with majorized scale and shape parameters separately, when the components are from two heterogeneous LL distributions as well as from the multiple outlier LL random variables. The rest of the paper is organized as follows. In Section 2, we have given the required notations, definitions and some useful lemmas which have been used throughout the paper. Results related to reversed hazard rate ordering and likelihood ratio ordering between two order statistics $X_{n:n}$ and $Y_{n:n}$ are derived in Section 3.

Throughout the paper, the word increasing (resp. decreasing) and nondecreasing (resp. nonincreasing) are used interchangeably, and $\Re$ denotes the set of real numbers $\{x : -\infty < x < \infty\}$. We also write $a \stackrel{sign}{=} b$ to mean that a and b have the same sign. For any differentiable function $k(\cdot)$, we write $k'(t)$ to denote the first derivative of $k(t)$ with respect to t.

2   Notations, Definitions and Preliminaries

For an absolutely continuous random variable X, we denote the probability density function, the distribution function and the reversed hazard rate function by $f_X(\cdot), F_X(\cdot)$, and $\tilde{r}_X(\cdot)$ respectively. The survival or reliability function of the random variable X is written as $\bar{F}_X(\cdot) = 1 - F_X(\cdot)$.

In order to compare different order statistics, stochastic orders are used for fair and reasonable comparison. In literature many different kinds of stochastic orders have been developed and studied. The following well known definitions may be obtained in Shaked and Shanthikumar [19].

Definition 2.1 *Let X and Y be two absolutely continuous random variables with respective supports (l_X, u_X) and (l_Y, u_Y), where u_X and u_Y may be positive infinity, and l_X and l_Y may be negative infinity. Then, X is said to be smaller than Y in*

(i) likelihood ratio (lr) order, denoted as $X \leq_{lr} Y$, if

$$\frac{f_Y(t)}{f_X(t)} \text{ is increasing in } t \in (l_X, u_X) \cup (l_Y, u_Y);$$

(ii) hazard rate (hr) order, denoted as $X \leq_{hr} Y$, if

$$\frac{\bar{F}_Y(t)}{\bar{F}_X(t)} \text{ is increasing in } t \in (-\infty, max(u_X, u_Y)),$$

which can equivalently be written as $r_X(t) \geq r_Y(t)$ for all t;

(iii) reversed hazard rate (rhr) order, denoted as $X \leq_{rhr} Y$, if

$$\frac{F_Y(t)}{F_X(t)} \text{ is increasing in } t \in (min(l_X, l_Y), \infty),$$

which can equivalently be written as $\tilde{r}_X(t) \leq \tilde{r}_Y(t)$ for all t;

(iv) usual stochastic (st) order, denoted as $X \leq_{st} Y$, if $\bar{F}_X(t) \leq \bar{F}_Y(t)$ for all $t \in (-\infty, \infty)$.

In the following diagram we present a chain of implications of the stochastic orders, see, for instance, Shaked and Shanthikumar [19], where the definitions and usefulness of these orders can be found.

$$X \leq_{hr} Y$$
$$\uparrow \qquad \searrow$$
$$X \leq_{lr} Y \quad \rightarrow \quad X \leq_{st} Y.$$
$$\downarrow \qquad \nearrow$$
$$X \leq_{rhr} Y$$

It is well known that the results on different stochastic orders can be established on using majorization order(s). Let I^n denotes an n-dimensional Euclidean space where $I \subseteq \Re$. Further, let $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in I^n$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in I^n$ be any two real vectors with $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ being the increasing arrangements of the components of the vector $\mathbf{x}$. The following definitions may be found in Marshall *et al.* [15].

Definition 2.2 *The vector $\mathbf{x}$ is said to majorize the vector $\mathbf{y}$ (written as $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$) if*

$$\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)}, \ j = 1, \ 2, \ \ldots, n - 1, \ \ and \ \ \sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}.$$

Definition 2.3 *A function $\psi : I^n \to \Re$ is said to be Schur-convex (resp. Schur-concave) on I^n if*

$$\mathbf{x} \overset{m}{\succeq} \mathbf{y} \text{ implies } \psi(\mathbf{x}) \geq (resp. \ \leq) \ \psi(\mathbf{y}) \ for \ all \ \mathbf{x}, \ \mathbf{y} \in I^n.$$

Notation 2.1 *Let us introduce the following notations.*

(i) $\mathcal{D}_+ = \{(x_1, x_2, \ldots, x_n) : x_1 \geq x_2 \geq \ldots \geq x_n > 0\}$.

(ii) $\mathcal{E}_+ = \{(x_1, x_2, \ldots, x_n) : 0 < x_1 \leq x_2 \leq \ldots \leq x_n\}$.

Next, two lemmas are given which will be used to prove our main results. The first one can be obtained by combining Proposition H2 of Marshall *et al.* ([15], p. 132) and Lemma 3.2 of Kundu *et al.* ([9]) while the second one is due to Lemma 3.4 of Kundu *et al.* ([9]).

Lemma 2.1 *Let $\varphi(\mathbf{x}) = \sum_{i=1}^{n} g_i(x_i)$ with $\mathbf{x} \in \mathcal{D}_+$, where $g_i : \mathbb{R} \to \mathbb{R}$ is differentiable, for all $i = 1, 2, \ldots, n$. Then $\varphi(\mathbf{x})$ is Schur-convex (Schur-concave) on $\mathcal{D}_+$ if, and only if,*

$$g_i'(a) \geq \ (resp. \leq) \ g_{i+1}'(b) \ whenever \ a \geq b, \ for \ all \ i = 1, 2, \ldots, n-1,$$

where $g'(a) = \frac{dg(x)}{dx}\big|_{x=a}$.

Lemma 2.2 *Let $\varphi(\mathbf{x}) = \sum_{i=1}^{n} g_i(x_i)$ with $\mathbf{x} \in \mathcal{E}_+$, where $g_i : \mathbb{R} \to \mathbb{R}$ is differentiable, for all $i = 1, 2, \ldots, n$. Then $\varphi(\mathbf{x})$ is Schur-convex (Schur-concave) on $\mathcal{E}_+$ if, and only if,*

$$g_{i+1}'(a) \geq \ (resp. \leq) \ g_i'(b) \ whenever \ a \geq b, \ for \ all \ i = 1, 2, \ldots, n-1,$$

where $g'(a) = \frac{dg(x)}{dx}\big|_{x=a}$.

3 Main Results

For $i = 1, 2, \ldots, n$, let X_i (resp. Y_i) be n independent nonnegative random variables following LL distribution as given in (1.1).

If $F_{n:n}(\cdot)$ and $G_{n:n}(\cdot)$ be the distribution functions of $X_{n:n}$ and $Y_{n:n}$ respectively, where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n)$, $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_n)$, then

$$F_{n:n}(x) = \prod_{i=1}^{n} \frac{x^{\sigma_i}(1 + \sigma_i(\lambda_i - \log x))}{1 + \lambda_i \sigma_i},$$

and

$$G_{n:n}(x) = \prod_{i=1}^{n} \frac{x^{\theta_i}(1 + \theta_i(\delta_i - \log x))}{1 + \delta_i \theta_i}.$$

Again, if $\tilde{r}_{n:n}^{X}$ and $\tilde{r}_{n:n}^{Y}$ are the reversed hazard rate functions of $X_{n:n}$ and $Y_{n:n}$ respectively, then

$$\tilde{r}_{n:n}^{X}(x) = \sum_{i=1}^{n} \frac{\sigma_i}{x}\left(1 - \frac{1}{1 + \sigma_i(\lambda_i - \log x)}\right), \tag{3.1}$$

and

$$\tilde{r}_{n:n}^{Y}(x) = \sum_{i=1}^{n} \frac{\theta_i}{x}\left(1 - \frac{1}{1 + \theta_i(\delta_i - \log x)}\right). \tag{3.2}$$

The following two theorems show that under certain conditions on parameters, there exists reversed hazard rate ordering between $X_{n:n}$ and $Y_{n:n}$.

Theorem 3.1 *For $i = 1, 2, \ldots, n$, let X_i and Y_i be two sets of mutually independent random variables with $X_i \sim LL(\sigma_i, \lambda_i)$ and $Y_i \sim LL(\theta_i, \lambda_i)$. Further, suppose that $\boldsymbol{\sigma}, \boldsymbol{\theta}, \boldsymbol{\lambda} \in \mathcal{D}_+$ or $\boldsymbol{\sigma}, \boldsymbol{\theta}, \boldsymbol{\lambda} \in \mathcal{E}_+$. Then,*

$$\boldsymbol{\sigma} \overset{m}{\succeq} \boldsymbol{\theta} \ implies \ X_{n:n} \geq_{rhr} Y_{n:n}.$$
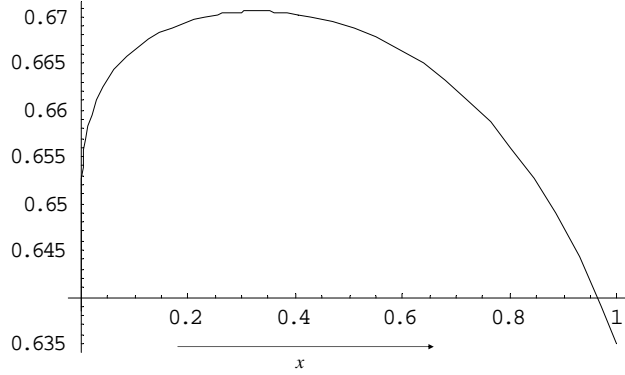
Figure 1: Graph of $\frac{F_{3:3}(x)}{G_{3:3}(x)}$

Proof: Let $g_i(y) = \frac{y}{x}\left(1 - \frac{1}{1+y(\lambda_i - \log x)}\right)$. Differentiating $g_i(y)$ with respect to y, we get

$$g_i'(y) = \frac{1}{x}\left(1 - \frac{1}{\left(1 + y\left(\lambda_i - \log x\right)\right)^2}\right),$$

giving

$$g_i'(\sigma_i) - g_{i+1}'(\sigma_{i+1}) = \frac{\left(1 + \sigma_i\left(\lambda_i - \log x\right)\right)^2 - \left(1 + \sigma_{i+1}\left(\lambda_{i+1} - \log x\right)\right)^2}{x\left(\left(1 + \sigma_i\left(\lambda_i - \log x\right)\right)\left(1 + \sigma_{i+1}\left(\lambda_{i+1} - \log x\right)\right)\right)^2}.$$

So, if $\boldsymbol{\sigma}, \boldsymbol{\lambda} \in \mathcal{D}_+$ (*resp.* $\mathcal{E}_+$), then $g_i'(\sigma_i) - g_{i+1}'(\sigma_{i+1}) \geq (\leq) 0$. Then, by Lemma 2.1 (Lemma 2.2), $\tilde{r}_{n:n}^X(x)$ is Schur convex in $\boldsymbol{\sigma}$, proving the result. $\qquad\square$

The counterexample given below shows that the ascending (descending) order of the components of the scale and shape parameters are necessary for the result of Theorem 3.1 to hold.

Counterexample 3.1 *Let $X_i \sim LL\left(\sigma_i, \lambda_i\right)$ and $Y_i \sim LL\left(\theta_i, \lambda_i\right), i = 1, 2, 3$. Now, if $(\sigma_1, \sigma_2, \sigma_3) = (1, 1, 5) \in \mathcal{E}_+$, $(\theta_1, \theta_2, \theta_3) = (1, 2, 4) \in \mathcal{E}_+$ and $(\lambda_1, \lambda_2, \lambda_3) = (4, 3, 0.2) \in \mathcal{D}_+$ are taken, then from Figure 1, it is clear that $\frac{F_{3:3}(x)}{G_{3:3}(x)}$ is not monotone, giving that $X_{3:3} \not\succeq_{rhr} Y_{3:3}$, although $\boldsymbol{\sigma} \overset{m}{\succeq} \boldsymbol{\theta}$.*

Theorem 3.1 guarantees that for parallel systems of components having independent LL distributed lifetimes with common scale parameter vector, the majorized shape parameter vector leads to larger system's life in the sense of the reversed hazard rate ordering. Now the question arises—what will happen if the scale parameter $\boldsymbol{\lambda}$ majorizes $\boldsymbol{\delta}$ when the shape parameter vector remains constant? The theorem given below answers that if the order of the components of shape and scale parameter vectors are reversed, then $X_{n:n}$ will be smaller than $Y_{n:n}$ in reversed hazard rate ordering.

Theorem 3.2 *For $i = 1, 2, \ldots, n$, let X_i and Y_i be two sets of mutually independent random variables with $X_i \sim LL(\sigma_i, \lambda_i)$ and $Y_i \sim LL(\sigma_i, \delta_i)$. Further, suppose that $\boldsymbol{\sigma} \in \mathcal{E}_+$, $\boldsymbol{\lambda}, \boldsymbol{\delta} \in \mathcal{D}_+$ or $\boldsymbol{\sigma} \in \mathcal{D}_+$, $\boldsymbol{\lambda}, \boldsymbol{\delta} \in \mathcal{E}_+$. Then,*

$$\boldsymbol{\lambda} \overset{m}{\succeq} \boldsymbol{\delta} \text{ implies } X_{n:n} \leq_{rhr} Y_{n:n}.$$

Proof: For $i = 1, 2 \ldots, n$, let us consider $g_i(y) = \frac{\sigma_i}{x}\left(1 - \frac{1}{1 + \sigma_i(y - \log x)}\right)$. Differentiating $g_i(y)$ with respect to y, we get

$$g_i'(y) = \frac{\sigma_i^2}{x\left(1 + \sigma_i\left(y - \log x\right)\right)^2},$$

giving

$$g_i'(\lambda_i) - g_{i+1}'(\lambda_{i+1}) \overset{sign}{=} \left(\sigma_i^2 - \sigma_{i+1}^2\right) + \sigma_i^2 \sigma_{i+1}^2 \left[(\lambda_{i+1} - \log x)^2 - (\lambda_i - \log x)^2\right]$$
$$+ 2\sigma_i \sigma_{i+1}\left[(\sigma_i \lambda_{i+1} - \sigma_{i+1}\lambda_i) - \log x\left(\sigma_i - \sigma_{i+1}\right)\right].$$

So, if $\boldsymbol{\lambda} \in \mathcal{D}_+$ (*resp. $\mathcal{E}_+$*) and $\boldsymbol{\sigma} \in \mathcal{E}_+$ (*resp. $\mathcal{D}_+$*), then $g_i'(\lambda_i) - g_{i+1}'(\lambda_{i+1}) \leq (\geq) 0$. So, by Lemma 2.1 (Lemma 2.2), $\tilde{r}_{n:n}^X(x)$ is Schur-concave in $\boldsymbol{\lambda}$, proving the result. $\qquad \square$

Next, one counterexample is provided to show that, nothing can be said about reversed hazard rate ordering between $X_{n:n}$ and $Y_{n:n}$ if $\boldsymbol{\lambda}$ majorizes $\boldsymbol{\delta}$ and all of $\boldsymbol{\lambda}$, $\boldsymbol{\delta}$ and $\boldsymbol{\sigma}$ are either in $\mathcal{E}_+$ or in $\mathcal{D}_+$.

Counterexample 3.2 *Let $X_i \sim LL(\sigma_i, \lambda_i)$ and $Y_i \sim (\sigma_i, \delta_i), i = 1, 2, 3$. Let $(\lambda_1, \lambda_2, \lambda_3) = (0.1, 0.3, 4.1) \in \mathcal{E}_+$ and $(\delta_1, \delta_2, \delta_3) = (0.2, 0.3, 4) \in \mathcal{E}_+$, giving $\boldsymbol{\lambda} \overset{m}{\succeq} \boldsymbol{\delta}$. Now, if $(\sigma_1, \sigma_2, \sigma_3) = (0.1, 3, 5) \in \mathcal{E}_+$ is taken, then Figure 2 (a) shows that $\frac{F_{3:3}(x)}{G_{3:3}(x)}$ is increasing in x. Again if $(\sigma_1, \sigma_2, \sigma_3) = (2, 3, 5) \in \mathcal{E}_+$ is taken, then Figure 2 (b) shows that $\frac{F_{3:3}(x)}{G_{3:3}(x)}$ is decreasing in x. So, it can be concluded that, for all $\boldsymbol{\sigma}, \boldsymbol{\lambda}, \boldsymbol{\delta} \in \mathcal{D}_+$(resp. $\mathcal{E}_+$), $\boldsymbol{\lambda} \overset{m}{\succeq} \boldsymbol{\delta}$ does not always imply $X_{3:3} \leq_{rhr} Y_{3:3}$.*

The following theorem shows that depending upon certain conditions, majorization order of the shape parameters implies likelihood ratio ordering between $X_{n:n}$ and $Y_{n:n}$.
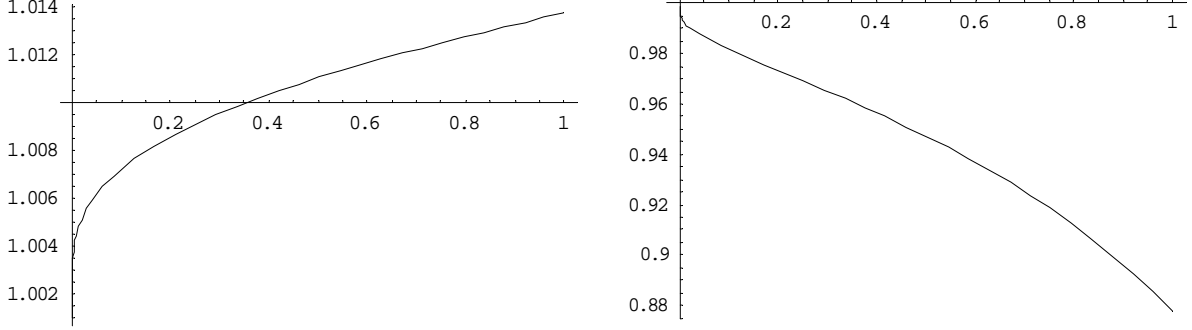
Theorem 3.3 *For $i = 1, 2, \ldots, n$, let X_i and Y_i be two sets of mutually independent random variables with $X_i \sim LL(\sigma_i, \lambda_i)$ and $Y_i \sim LL(\theta_i, \lambda_i)$. Further, suppose that $\boldsymbol{\sigma}, \boldsymbol{\theta}, \boldsymbol{\lambda} \in \mathcal{D}_+$ or $\boldsymbol{\sigma}, \boldsymbol{\theta}, \boldsymbol{\lambda} \in \mathcal{E}_+$. Then, if $\lambda_i \sigma_i > 1/2$,*

$$\boldsymbol{\sigma} \overset{m}{\succeq} \boldsymbol{\theta} \text{ implies } X_{n:n} \geq_{lr} Y_{n:n}.$$

Proof:

In view of theorem 3.1 and using (3.1) and (3.2), here we have only to show that

$$\frac{\tilde{r}_{n:n}^X(x)}{\tilde{r}_{n:n}^Y(x)} = \frac{\sum_{k=1}^n u_k(\sigma_k, x)}{\sum_{k=1}^n u_k(\theta_k, x)}$$
$$= \eta(x)(say),$$

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(a) For $(\sigma_1, \sigma_2, \sigma_3) = (0.1, 3, 5)$ (b) For $(\sigma_1, \sigma_2, \sigma_3) = (2, 3, 5)$

Figure 2: Graph of $\frac{F_{3:3}(x)}{G_{3:3}(x)}$

is increasing in x, where $u_k(y, x) = \frac{y^2(\lambda_k - \log x)}{1 + y(\lambda_k - \log x)}$. Now, differentiating $\eta(x)$ with respect to x,

$$\eta'(x) \overset{sign}{=} \sum_{k=1}^{n} \frac{\partial u_k(\sigma_k, x)}{\partial x} \sum_{k=1}^{n} u_k(\theta_k, x) - \sum_{k=1}^{n} \frac{\partial u_k(\theta_k, x)}{\partial x} \sum_{k=1}^{n} u_k(\sigma_k, x)$$

$$= -h(\boldsymbol{\sigma}, x) \sum_{k=1}^{n} u_k(\theta_k, x) + h(\boldsymbol{\theta}, x) \sum_{k=1}^{n} u_k(\sigma_k, x),$$

where

$$h(\boldsymbol{\sigma}, x) = -\sum_{k=1}^{n} \frac{\partial u_k(\sigma_k, x)}{\partial x} = \frac{1}{x} \sum_{k=1}^{n} \frac{\sigma_k^2}{(1 + \sigma_k(\lambda_k - \log x))^2}$$

and

$$h(\boldsymbol{\theta}, x) = -\sum_{k=1}^{n} \frac{\partial u_k(\theta_k, x)}{\partial x} = \frac{1}{x} \sum_{k=1}^{n} \frac{\theta_k^2}{(1 + \theta_k(\lambda_k - \log x))^2}.$$

Thus, to show that $\eta(x)$ is increasing in x, we have only to show that

$$\psi(\boldsymbol{\sigma}, x) = \frac{h(\boldsymbol{\sigma}, x)}{\sum_{k=1}^{n} u_k(\sigma_k, x)}$$

is Schur-concave in $\boldsymbol{\sigma}$.

Now, as

$$\frac{\partial h(\boldsymbol{\sigma}, x)}{\partial \sigma_i} = \frac{1}{x} \cdot \frac{2\sigma_i}{(1 + \sigma_i(\lambda_i - \log x))^3}$$

and

$$\frac{\partial}{\partial \sigma_i} \left[\sum_{k=1}^{n} u_k(\sigma_k, x) \right] = 1 - \frac{1}{(1 + \sigma_i(\lambda_i - \log x))^2},$$

then

$$\frac{\partial \psi}{\partial \sigma_i} \overset{sign}{=} \frac{2\sigma_i}{(1 + \sigma_i(\lambda_i - \log x))^3} \sum_{k=1}^{n} u_k(\sigma_k, x) - x.h(\boldsymbol{\sigma}, x) \left(1 - \frac{1}{(1 + \sigma_i(\lambda_i - \log x))^2}\right).$$

So, if $\boldsymbol{\sigma}, \boldsymbol{\lambda} \in \mathcal{D}_+ \, (resp \in \mathcal{E}_+)$, i.e., for $i \leq j$ if $\sigma_i \geq \sigma_j$ and $\lambda_i \geq \lambda_j \, (\sigma_i \leq \sigma_j, \lambda_i \leq \lambda_j)$, then noticing the fact that $\frac{1}{(1+\sigma_i(\lambda_i - \log x))^2}$ is decreasing in σ_i as well as in λ_i, it can be written that

$$\frac{1}{(1 + \sigma_i(\lambda_i - \log x))^2} \leq (\geq) \frac{1}{(1 + \sigma_j(\lambda_i - \log x))^2} \leq (\geq) \frac{1}{(1 + \sigma_j(\lambda_j - \log x))^2}.$$

Again, as $\sigma_i \lambda_i > \frac{1}{2}$ implying $\sigma_i(\lambda_i - \log x) > \frac{1}{2}$ for all $0 < x < 1$, then

$$\frac{\partial}{\partial \sigma_i}\left(\frac{\sigma_i}{(1 + \sigma_i(\lambda_i - \log x))^3}\right) = \frac{1 - 2\sigma_i(\lambda_i - \log x)}{(1 + \sigma_i(\lambda_i - \log x))^4} < 0,$$

proving that $\frac{\sigma_i}{(1+\sigma_i(\lambda_i - \log x))^3}$ is decreasing in σ_i. Again, it is also decreasing in λ_i. Thus, for all $\sigma_i \geq \sigma_j$ and $\lambda_i \geq \lambda_j \, (\sigma_i \leq \sigma_j, \lambda_i \leq \lambda_j)$,

$$\frac{\sigma_i}{(1 + \sigma_i(\lambda_i - \log x))^3} \leq (\geq) \frac{\sigma_j}{(1 + \sigma_j(\lambda_i - \log x))^3} \leq (\geq) \frac{\sigma_j}{(1 + \sigma_j(\lambda_j - \log x))^3}.$$

So, for all $i \leq j$

$$\frac{\partial \psi}{\partial \sigma_i} - \frac{\partial \psi}{\partial \sigma_j} \overset{sign}{=} \sum_{k=1}^{n} \frac{\sigma_k^2(\lambda_k - \log x)}{1 + \sigma_k(\lambda_k - \log x)} \left[\frac{2\sigma_i}{(1 + \sigma_i(\lambda_i - \log x))^3} - \frac{2\sigma_j}{(1 + \sigma_j(\lambda_j - \log x))^3}\right]$$
$$+ \sum_{k=1}^{n} \frac{\sigma_k^2}{(1 + \sigma_k(\lambda_k - \log x))^2} \left[\frac{1}{(1 + \sigma_i(\lambda_i - \log x))^2} - \frac{1}{(1 + \sigma_j(\lambda_j - \log x))^2}\right]$$
$$\leq (\geq) 0.$$

Thus the result follows from Lemma 3.1 (Lemma 3.3) of Kundu *et al.* ([9]). □

Although Theorem 3.3 holds under a sufficient condition for two n component systems, the next theorem shows that no such condition is required for these systems having multiple-outlier LL model if the scale parameter vectors of these systems are common.

Theorem 3.4 *For $i = 1, 2, ..., n$, let X_i and Y_i be two sets of independent random variables each following the multiple-outlier EW model such that $X_i \sim LL(\sigma, \lambda)$ and $Y_i \sim LL(\theta, \lambda)$ for $i = 1, 2, \ldots, n_1$, $X_i \sim LL(\sigma^*, \lambda^*)$ and $Y_i \sim LL(\theta^*, \lambda^*)$ for $i = n_1 + 1, n_1 + 2, \ldots, n_1 + n_2(= n)$ If*

$$(\underbrace{\sigma, \sigma, \ldots, \sigma}_{n_1}, \underbrace{\sigma^*, \sigma^*, \ldots, \sigma^*}_{n_2}) \overset{m}{\succeq} (\underbrace{\theta, \theta, \ldots, \theta}_{n_1}, \underbrace{\theta^*, \theta^*, \ldots, \theta^*}_{n_2})$$

and either $\{\sigma \geq \sigma^*, \theta \geq \theta^*, \lambda \geq \lambda^*\}$ or $\{\sigma \leq \sigma^*, \theta \leq \theta^*, \lambda \leq \lambda^*\}$ then $X_{n:n} \geq_{lr} Y_{n:n}$.

Proof: Following Theorem 3.3 and in view of Theorem 3.1, we have only to show that

$$\psi_1(\boldsymbol{\sigma}, x) = \frac{\sum_{k=1}^{n} \frac{\sigma_k^2}{(1+\sigma_k(\lambda_k - \log x))^2}}{\sum_{k=1}^{n} \frac{\sigma_k^2(\lambda_k - \log x)}{1+\sigma_k(\lambda_k - \log x)}}$$

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is Schur-concave in $\boldsymbol{\sigma}$.

Now, three cases may arise:

$Case(i)$ If $1 \leq i < j \leq n_1$, i.e., if $\sigma_i = \sigma_j = \sigma$ and $\lambda_i = \lambda_j = \lambda$, then $\frac{\partial \psi_1}{\partial \sigma_i} - \frac{\partial \psi_1}{\partial \sigma_j} = 0$.

$Case(ii)$ If $n_1 + 1 \leq i < j \leq n$, i.e., if $\sigma_i = \sigma_j = \sigma^*$ and $\lambda_i = \lambda_j = \lambda^*$, then $\frac{\partial \Psi}{\partial \sigma_i} - \frac{\partial \Psi}{\partial \sigma_j} = 0$.

$Case(iii)$ If $1 \leq i \leq n_1$ and $n_1 + 1 \leq j \leq n$, then $\sigma_i = \sigma$, $\lambda_i = \lambda$ and $\sigma_j = \sigma^*$, $\lambda_i = \lambda^*$. It can be easily shown that

$$\frac{\partial \psi_1}{\partial \sigma_i} - \frac{\partial \psi_1}{\partial \sigma_j} \overset{sign}{=} \left(\frac{n_1 \sigma^2}{(1+\xi_1)^2} + \frac{n_2 \sigma^{*2}}{(1+\xi_2)^2} \right) \left(\frac{\xi_2^2}{(1+\xi_2)^2} - \frac{\xi_1^2}{(1+\xi_1)^2} \right)$$
$$+ \left(\frac{\sigma \xi_2}{(1+\xi_1)} - \frac{\sigma^* \xi_1}{(1+\xi_2)} \right) \left(\frac{2n_1 \sigma}{(1+\xi_2)^2(1+\xi_1)} + \frac{2n_2 \sigma^*}{(1+\xi_1)^2(1+\xi_2)} \right).$$

where $\xi_1 = \sigma(\lambda - \log x)$ and $\xi_2 = \sigma^*(\lambda^* - \log x)$. Now, as $\sigma \geq (\leq)\sigma^*$ and $\lambda \geq (\leq)\lambda^*$, implying that $\sigma(\lambda - \log x) \geq (\leq)\sigma^*(\lambda^* - \log x)$ i.e. $\xi_1 \geq (\leq)\xi_2$, and moreover, $\frac{\xi}{1+\xi} = 1 - \frac{1}{1+\xi}$ is increasing in ξ, then $\frac{\xi_2^2}{(1+\xi_2)^2} \leq (\geq) \frac{\xi_1^2}{(1+\xi_1)^2}$. Again,

$$\frac{\sigma \xi_2}{1+\xi_1} - \frac{\sigma^* \xi_1}{1+\xi_2} = \frac{\sigma \sigma^* \{(\lambda^* - \log x)(1 + \sigma^*(\lambda^* - \log x)) - (\lambda - \log x)(1 + \sigma(\lambda - \log x))\}}{(1+\xi_1)(1+\xi_2)}$$
$$\leq (\geq) \quad 0.$$

So, by Lemma 3.1 (Lemma 3.3) of Kundu et $al.$ ([9]), the result is proved. □

Theorem 3.3 guarantees that, for two n component parallel systems (with a sufficient condition) having independent LL distributed lifetimes with a common scale parameter vector, the majorized shape parameter vector leads to greater system's lifetime in the sense of likelihood ratio order. The next theorem states that the majorized scale parameter vector leads to smaller system's lifetime in the sense of likelihood ratio order when the shape parameter vector of these two n-component parallel systems are common.

Theorem 3.5 For $i = 1, 2, \ldots, n$, let X_i and Y_i be two $sets$ of $mutually$ $independent$ $random$ $variables$ $with$ $X_i \sim LL(\sigma_i, \lambda_i)$ and $Y_i \sim LL(\sigma_i, \delta_i)$. $Further$, $suppose$ $that$ $\boldsymbol{\sigma} \in \mathcal{E}_+$, $\boldsymbol{\lambda}, \boldsymbol{\delta} \in \mathcal{D}_+$ or $\boldsymbol{\sigma} \in \mathcal{D}_+$, $\boldsymbol{\lambda}, \boldsymbol{\delta} \in \mathcal{E}_+$. $Then$,

$$\boldsymbol{\lambda} \overset{m}{\succeq} \boldsymbol{\delta} \ implies \ X_{n:n} \leq_{lr} Y_{n:n}.$$

Proof: In view of Theorem 3.2 and using (3.1) and (3.2), we are to prove that

$$\eta_1(x) = \frac{\sum_{k=1}^{n} \frac{\sigma_k}{x} \left(1 - \frac{1}{1 + \sigma_k(\lambda_k - \log x)} \right)}{\sum_{k=1}^{n} \frac{\sigma_k}{x} \left(1 - \frac{1}{1 + \sigma_k(\delta_k - \log x)} \right)}$$

is decreasing in x $i.e.$ to prove that

$$\psi_2(\boldsymbol{\lambda}, x) = \frac{\sum_{k=1}^{n} \frac{\sigma_k^2}{(1 + \sigma_k(\lambda_k - \log x))^2}}{\sum_{k=1}^{n} \frac{\sigma_k^2(\lambda_k - \log x)}{(1 + \sigma_k(\lambda_k - \log x))}}$$

is Schur-convex in $\boldsymbol{\lambda}$. Now,

$$\frac{\partial \psi_2}{\partial \lambda_i} \stackrel{sign}{=} -\frac{2\sigma_i^3}{\left(1 + \sigma_i\left(\lambda_i - \log x\right)\right)^3} \sum_{k=1}^{n} \frac{\sigma_k^2\left(\lambda_k - \log x\right)}{\left(1 + \sigma_k\left(\lambda_k - \log x\right)\right)} - \frac{\sigma_i^2}{\left(1 + \sigma_i\left(\lambda_i - \log x\right)\right)^2} \sum_{k=1}^{n} \frac{\sigma_k^2}{\left(1 + \sigma_k\left(\lambda_k - \log x\right)\right)^2}.$$

So, by noticing the fact that

$$\frac{\partial}{\partial \sigma_i}\left[\frac{\sigma_i}{\left(1 + \sigma_i\left(\lambda_i - \log x\right)\right)}\right] = \frac{1}{\left(1 + \sigma_i\left(\lambda_i - \log x\right)\right)^2} > 0,$$

giving that $\frac{\sigma_i}{(1+\sigma_i(\lambda_i - \log x))}$ is increasing in σ_i, $\boldsymbol{\lambda} \in \mathcal{D}_+ \left(resp. \mathcal{E}_+\right)$ and $\boldsymbol{\sigma} \in \mathcal{E}_+ \left(resp. \mathcal{D}_+\right)$, $i.e.$ for all $i \leq j$ $\lambda_i \geq (\leq)\lambda_j$ and $\sigma_i \leq (\geq)\sigma_j$ gives

$$\frac{\sigma_i^3}{\left(1 + \sigma_i\left(\lambda_i - \log x\right)\right)^3} \leq (\geq)\frac{\sigma_j^3}{\left(1 + \sigma_j\left(\lambda_i - \log x\right)\right)^3} \leq (\geq)\frac{\sigma_j^3}{\left(1 + \sigma_j\left(\lambda_j - \log x\right)\right)^3}$$

and

$$\frac{\sigma_i^2}{\left(1 + \sigma_i\left(\lambda_i - \log x\right)\right)^2} \leq (\geq)\frac{\sigma_j^2}{\left(1 + \sigma_j\left(\lambda_i - \log x\right)\right)^2} \leq (\geq)\frac{\sigma_j^2}{\left(1 + \sigma_j\left(\lambda_j - \log x\right)\right)^2}.$$

So,

$$\frac{\partial \psi_2}{\partial \lambda_i} - \frac{\partial \psi_2}{\partial \lambda_j} \geq (\leq)0.$$

Thus the result follows from Lemma 3.1 (Lemma 3.3) of Kundu et $al.$ ([9]). $\qquad \square$

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