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great by
deeds, not by
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**A simpler algorithm to price American Lookback options in a
discrete stochastic volatility model**

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Abstract

This article develops an efficient pricing algorithm for American lookback options on a binomial lattice with stochastic volatility. This is achieved by combining the structure of this lattice together with the fact that one can price lookbacks on a standard binomial lattice without having to store the path variables. We apply this algorithm to study the efficiency of fractional lookback contracts, which are used as a benchmark for designing equity indexed annuities, and illustrate the impact of volatility persistence on their prices. This algorithm also extends the usefulness of the stochastic volatility model proposed by Aingworth, Das and Motwani (2006) by enabling the pricing of lookback options on their lattice.

JEL: G13

Key words: American Lookback Options, Stochastic Volatility, Binomial lattice, Markov switching, Volatility persistence

1 Introduction

Dynamic programming enables the exact pricing of american options in discrete models through a simple backward recursion. This idea is extensively used to get approximate prices of American options, even under continuous time models. However pricing of american style exotic options has always posed a challenge, both for discrete and continuous time models.

Though backward recursion can be used in discrete time models to price such options, the challenge comes from the fact that along with the stock price, one also needs to keep track of an additional state variable (usually a path dependent variable capturing the past) at every node. For example, for a floating strike lookback put option, this additional variable will be the running maximum of the previous prices until that node, which is highly path dependent. Similarly for an Asian option, we need to keep track of the running average of the past prices.

For more complex exotics, more than one additional state variable may be needed to get the exact price through backward recursion. Keeping track of additional variables adds considerably to the computational burden of the pricing algorithm and makes exact computations infeasible, even for moderately large trees.

In this context, Hull and White (1993) suggested an exact algorithm for pricing American style lookback options and an approximate algorithm for American style Asian options, using the Cox Ross and Rubinstein (CRR) parametrization of the binomial model (Cox et al. (1979)). However their algorithm had the drawback of tracking the state variables at every node, in addition to the current stock price.

Cheuk and Vorst (1997) made use of homogeneity considerations and overcame this drawback by giving a simpler algorithm for pricing lookback options, which did not require the tracking of the additional path dependent variables. This was a significant step as it reduced the complexity of pricing lookbacks on binomial trees, down to that of pricing plain vanilla options.

Meanwhile Aingworth et al. (2006) proposed a simpler stochastic volatility model (referred to as the ADM model in the rest of the article) on the lattice, which considerably reduced the computational complexity arising in fully stochastic volatility models like Heston (1993). Their primary purpose of proposing a lattice model was to price American options. The main advantage of this model is it can accommodate multiple volatility regimes and with varying persistence levels for each regime. This model easily generates the volatility smiles and the volatility skews that are observed with market data and also has the advantage that it can be quickly calibrated to the observed option prices.

The ADM model amounts to tracking the joint process of the stock price and the volatility on a binomial lattice. In the simplest case of two possible levels of volatility, the binomial tree will have four branches emanating from each node, one for each combination of stock price going up or down and the volatility remaining at the current level or switching to the other level. For this model the authors developed an algorithm for pricing vanilla European and American options and

proved that its run time is only of polynomial order, unlike other closely related stochastic volatility lattice models where the run time is exponential. They achieved this by discovering a recombining structure of the lattice, which is not present in some of the related stochastic volatility models.

However pricing Exotic options on a stochastic volatility lattice is going to be a much more complex computational task due to the storage of additional path dependent variables. Traversing all possible paths for a n period tree that takes into account stochastic volatility, represents a prohibitive computational challenge, even for small values of n .

Many authors since Naik (1993) have focused on pricing vanilla options in models, where the parameters governing the underlying asset process follow a regime-switching framework, similar to the ADM model. The reader can find a recent summary of these developments in Costabile et al. (2014). For this study, which explores the pricing of exotic options under such models, we have chosen the ADM model for its simplicity. Here we show how to overcome the computational challenge in the specific case of pricing lookback options.

The initial motivation for this article was to check whether the computations for pricing exotic options can be brought down to polynomial time complexity. In the particular case of pricing lookbacks, instead of keeping track of the additional state variable, such as the running maximum or the minimum, we were interested in extending the Cheuk and Vorst approach that exploits the homogeneity considerations. Such an extension would have two major advantages. First it would have the same polynomial time complexity as that of the algorithm for pricing vanilla American options. Second, it could eliminate the storage of additional path dependent variables, thereby significantly reducing the computational burden as compared to standard Hull and White-type algorithms. This is the objective of the current study.

Finally, the simplicity of lookback options is one of the main reasons for its popularity among exotic options (see for example Heynen and Kat (1997) and Gerber and Shiu (2003)). In the words of Conze and Viswanathan (1991),

... among these, lookback options are very attractive options because they keep track of past events. As a consequence, they allow its holder to take advantage of anticipated market movements without knowing the exact date of their occurrences, and may provide psychological comfort by minimizing regrets.

While several studies have discussed the pricing of American lookbacks in the Black-Scholes framework (see for example Kimura (2011), Lai and Lim (2004) and references therein), our paper

focuses on a stochastic volatility model and adds to the available pricing algorithms for American lookbacks in this setup.

As an application of the pricing algorithm developed here, prices of the closely related fractional lookback options, also known as partial lookbacks, are also reported. The fractional lookback is a cheaper variant of the lookback, with payoffs given by,

$$\left(S(T) - \alpha \min_{0 \leq i \leq N} S(t_i)\right)_+ \quad \text{and} \quad \left(\beta \max_{0 \leq i \leq N} S(t_i) - S(T)\right)_+$$

where typically $\alpha > 1$ and $\beta < 1$, while $S(t)$ denotes the underlying asset price at time t (the notations are described in more detail in the following section). Such lookback options serve as an useful benchmark while pricing Equity Indexed Annuities that offer investors participation in the equity markets with a principal protection feature (Tiong (2000)).

The rest of the paper is organized as follows. The next section describes the stochastic volatility lattice and the algorithm for pricing floating strike lookback calls. Section 3 discusses the extensions related to observation frequency and early exercise. Numerical results are reported in section 4 and section 5 concludes.

2 Stochastic Volatility lattice

We give here a quick overview of the ADM model and use the same notation to describe our algorithm. For illustration the case of pricing currency options is considered. The same treatment with slight modifications can be applied for options on other assets.

In the standard CRR parametrization of the Binomial model, exchange rates evolve as follows at time points t_0, t_1, \dots, t_N : $S(0) = S_0$ and for $i \geq 0$, $S(t_{i+1})$ is either $S(t_i)u$ or $S(t_i)d$, where u and d are the returns for the up and down moves respectively. The time points are typically equispaced, i.e. $t_i = \frac{iT}{N}$, where T is the length of the time horizon and N is the number of periods in the lattice. The up and down returns are given by $u = \exp(\sigma\sqrt{T/N})$ and $d = 1/u$, where σ is the exchange rate volatility that is assumed to be constant for the entire time horizon.

The ADM model, which is a stochastic volatility version of this basic model, assumes that σ can take multiple values namely σ_j , $j = 1, \dots, k$ and changes in the volatility are governed by a Markov chain. Here at every time point, there is a joint evolution of both the exchange rate and the volatility in the following sense: the exchange rate process goes up or down (the magnitude of which depends on the current level of volatility) and the volatility process can remain at the same

level or switch to a different level. The model specification could also include a correlation between the volatility process and the exchange rates.

The main challenge with this lattice is its recombining nature. As mentioned earlier, even with two volatility states, each node in the tree will give rise to four other nodes for the next time period, one for each combination of exchange rate going up/down and volatility staying the same/switches to the other state. The recombining is not very clear, as an up move to the low volatility state followed by a down move to the high volatility state will result in a different price if the sequence is reversed. Unless this is resolved, we will have the difficulty that at time period N , we will have to deal with 4^N nodes and the computing efforts will be prohibitive even for pricing a vanilla option at moderate values of N . Aingworth et al. (2006) resolved this issue by identifying a recombining structure within this lattice, which we use in our algorithm.

We restrict our attention to the case of two volatility levels (a “normal” and a “high” volatility regime), to keep the exposition simple. The transition matrix for the markov chain governing the volatility switches is denoted as:

$$P = \begin{bmatrix} p_L & 1 - p_L \\ 1 - p_H & p_H \end{bmatrix}.$$

With this notation, p_L and p_H also reflects the degree of persistence in the low and high volatility states. Corresponding to the two volatility levels, which from here on will be denoted as σ_L and σ_H , we have u_L, d_L, u_H and d_H defined in the usual manner, as in the beginning of this section.

When $\sigma(t_i) = \sigma_L$, the risk-neutral joint evolution of $(S(t), \sigma(t))$ is given by,

$$(S(t_{i+1}), \sigma(t_{i+1})) = \begin{cases} (S(t_i)u_L, \sigma_L) & \text{w/prob } q_L p_L \\ (S(t_i)d_L, \sigma_L) & \text{w/prob } (1 - q_L) p_L \\ (S(t_i)u_L, \sigma_H) & \text{w/prob } q_L (1 - p_L) \\ (S(t_i)d_L, \sigma_H) & \text{w/prob } (1 - q_L) (1 - p_L) \end{cases},$$

where

$$q_L = \frac{e^{(r_d - r_f)T/N} - d_L}{u_L - d_L},$$

r_d is the continuous-time risk-free domestic rate and r_f is the corresponding risk-free foreign rate. Analogously when $\sigma(t_i) = \sigma_H$, we replace u_L, d_L and q_L with u_H, d_H and q_H respectively and adjust

the transition probabilities as below:

$$(S(t_{i+1}), \sigma(t_{i+1})) = \begin{cases} (S(t_i)u_H, \sigma_H) & \text{w/prob } q_H p_H \\ (S(t_i)d_H, \sigma_H) & \text{w/prob } (1 - q_H) p_H \\ (S(t_i)u_H, \sigma_L) & \text{w/prob } q_H (1 - p_H) \\ (S(t_i)d_H, \sigma_L) & \text{w/prob } (1 - q_H) (1 - p_H) \end{cases},$$

We first consider the pricing of an European floating strike lookback call option, whose payoff at maturity is given by,

$$S(T) - \min_{0 \leq i \leq N} S(t_i).$$

For such an option, the option value at any intermediate time t_j , will depend not only on the exchange rate at that time point, $S(t_j)$, but also the running minimum,

$$m(t_j) = \min_{0 \leq i \leq j} S(t_i).$$

In order to value this option at inception, the usual method of going backwards on the lattice is followed. But we will do this in a manner similar to the Cheuk and Vorst algorithm, so as to avoid keeping track of the running minimum directly at every node.

At time t_j , let k_1 be the integer that captures the difference in powers of u_L between $S(t_j)$ and $m(t_j)$ and similarly define k_2 for the difference in powers of u_H . Thus,

$$S(t_j) = m(t_j) u_L^{k_1} u_H^{k_2}.$$

We now prove the following claim, which will simplify our option pricing algorithm and in turn significantly reduce the computational efforts.

Claim: If $C(S(t_j), m(t_j), t_j, \sigma(t_j))$ denotes the intermediate value of the option at time t_j , then it can be written as,

$$C(S(t_j), m(t_j), t_j, \sigma(t_j)) = S(t_j) V(k_1, k_2, t_j, \sigma(t_j)),$$

for some function V depending only on k_1, k_2, t_j and $\sigma(t_j)$.

We now prove the claim through backward induction. At maturity,

$$C(S(t_N), m(t_N), t_N, \sigma(t_N)) = S(t_N) (1 - u_L^{-k_1} u_H^{-k_2}),$$

for both the values of $\sigma(t_N)$. So the claim holds at maturity if we define,

$$V(k_1, k_2, t_N, \sigma(t_N)) = 1 - u_L^{-k_1} u_H^{-k_2}.$$

Now to prove the claim for other time periods, consider the following three cases, starting with the simpler ones. We will assume the claim to be true for time t_{j+1} and use this to prove it for t_j .

Case A: $k_1 \geq 1$ and $k_2 \geq 1$.

Without loss of generality, let $\sigma(t_j) = \sigma_L$. If the underlying exchange rate goes from $S(t_j)$ to $S(t_j)u_L$, then the difference in powers in u_L , (i.e. k_1) increases by 1 and in a down move k_1 decreases by 1. In all this k_2 remains unaffected. The value of the lookback at t_j using risk-neutral pricing is given by,

$$C(S(t_j), m(t_j), t_j, \sigma(t_j) = \sigma_L) = \frac{1}{r} \left\{ q_L p_L C(S(t_j)u_L, m(t_{j+1}), t_{j+1}, \sigma_L) + (1 - q_L) p_L C(S(t_j)d_L, m(t_{j+1}), t_{j+1}, \sigma_L) + q_L (1 - p_L) C(S(t_j)u_L, m(t_{j+1}), t_{j+1}, \sigma_H) + (1 - q_L) (1 - p_L) C(S(t_j)d_L, m(t_{j+1}), t_{j+1}, \sigma_H) \right\},$$

where $r = e^{raT/N}$. Combining the observation on k_1 and the backward induction hypothesis, the right hand side can be written as,

$$\frac{1}{r} \left\{ q_L p_L S(t_j)u_L V(k_1 + 1, k_2, t_{j+1}, \sigma_L) + (1 - q_L) p_L S(t_j)d_L V(k_1 - 1, k_2, t_{j+1}, \sigma_L) + q_L (1 - p_L) S(t_j)u_L V(k_1 + 1, k_2, t_{j+1}, \sigma_H) + (1 - q_L) (1 - p_L) S(t_j)d_L V(k_1 - 1, k_2, t_{j+1}, \sigma_H) \right\}.$$

Taking $S(t_j)$ outside the braces in the above expression and defining whatever is remaining as $V(k_1, k_2, t_j, \sigma_L)$, completes the proof of the claim for this case.

Similarly for $\sigma(t_j) = \sigma_H$, the above expression will change to,

$$\frac{1}{r} \left\{ q_H p_H S(t_j)u_H V(k_1, k_2 + 1, t_{j+1}, \sigma_H) + (1 - q_H) p_H S(t_j)d_H V(k_1, k_2 - 1, t_{j+1}, \sigma_H) + q_H (1 - p_H) S(t_j)u_H V(k_1, k_2 + 1, t_{j+1}, \sigma_L) + (1 - q_H) (1 - p_H) S(t_j)d_H V(k_1, k_2 - 1, t_{j+1}, \sigma_L) \right\}.$$

Now defining the V function in a similar manner as above completes the proof.

Case B: $k_1 = k_2 = 0$.

In this case, the exchange rate right now is equal to the minimum reached so far and hence the difference in powers is zero. Again without loss of generality, let $\sigma(t_j) = \sigma_L$. Now if the underlying exchange rate goes from $S(t_j)$ to $S(t_j)u_L$, then k_1 increases to 1, while k_2 remains at 0. But if there is a downward move, then a new minimum is reached and both k_1 and k_2 remains at zero.

So, the definition of the V function for this case will be

$$\frac{1}{r} \left\{ q_L p_L u_L V(1, 0, t_{j+1}, \sigma_L) + (1 - q_L) p_L d_L V(0, 0, t_{j+1}, \sigma_L) + q_L (1 - p_L) u_L V(1, 0, t_{j+1}, \sigma_H) + (1 - q_L) (1 - p_L) d_L V(0, 0, t_{j+1}, \sigma_H) \right\}.$$

And for $\sigma(t_j) = \sigma_H$, this definition changes to,

$$\frac{1}{r} \left\{ q_H p_H u_H V(0, 1, t_{j+1}, \sigma_H) + (1 - q_H) p_H d_H V(0, 0, t_{j+1}, \sigma_H) \right. \\ \left. + q_H (1 - p_H) u_H V(0, 1, t_{j+1}, \sigma_L) + (1 - q_H) (1 - p_H) d_H V(0, 0, t_{j+1}, \sigma_L) \right\}.$$

This completes the proof for Case B.

Case C: One of k_1, k_2 is ≤ 0 and the other is ≥ 1 .

This case does not follow a natural generalization as the previous two cases, as shown in the illustration below. Before we proceed note that both k_1 and k_2 cannot both be less than zero, as we are computing the differences in the powers with the minimum exchange rate seen so far.

Suppose for example, $k_1 = 0, k_2 = 3$ and $\sigma(t_j) = \sigma_L$ currently. Let $S(t_j) = S_0 u_L^{-2} u_H^2$. This implies,

$$m(t_j) = S_0 u_L^{-2} u_H^{-1} \quad (= S_0 d_L^2 d_H).$$

Now from t_j to t_{j+1} , if the currency moves downward, then

$$S(t_{j+1}) = S_0 u_L^{-3} u_H^2.$$

The given information is insufficient to determine whether the currency has reached a new minimum or not, as this will depend on the relation between u_L and u_H . In order to achieve this we use the following rule :

Define $\xi = \frac{\log u_L}{\log u_H}$. A new minimum is reached if $(k_1 - 1)\xi + k_2 \leq 0$.

Continuing our example, suppose we know that $u_H = u_L^2$. This implies $\xi = 1/2$. So if $S(t_j) = S_0 u_L^{-2} u_H^2$, then after five successive down moves, all in the low volatility regime,

$$S(t_{j+5}) = S_0 u_L^{-7} u_H^2 \quad (\text{same as } S_0 u_L^{-1} u_H^{-1}).$$

This is still higher than the minimum. Now with one more down move in the low volatility regime, we reach the minimum, i.e. with $k_1 = -5$ and $k_2 = 3$, we will have $(k_1 - 1)\xi + k_2 = 0$.

The above example illustrates the following updating rule for the up and down moves when $\sigma(t_j) = \sigma_L$: if the underlying exchange rate goes from $S(t_j)$ to $S(t_j)u_L$, then k_1 increases by 1 and k_2 remains the same. Whereas if it is a down move, then k_1 decreases by 1 only when $(k_1 - 1)\xi + k_2 > 0$; otherwise, both k_1 and k_2 are set to 0, indicating that a new minimum is reached.

Similarly, if $k_2 \leq 0$ and $\sigma(t_j) = \sigma_H$ at any node, we follow the updating rule as given below: with a up move, k_2 increases by 1 and k_1 remains the same. For a down move, k_2 decreases by 1

only when $k_1\xi + (k_2 - 1) > 0$; otherwise, both k_1 and k_2 are set to 0, indicating that a new minimum is reached.

In Figure 1, we give a sample path in the lattice that illustrates the updating rule, where k_2 happens to take negative values. Here we have assumed that $u_H = u_L^{1.5}$, implying $\xi = 2/3$. For simplicity, we have suppressed the time variable and shown only the changes in the values of (k_1, k_2) and whether the node is in the low or high volatility state (denoted as L or H).

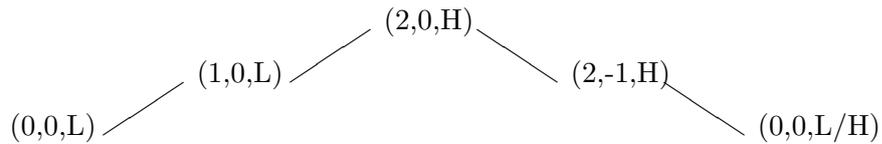


Figure 1: A sample branch illustrating the updating rule

Finally similar to cases A and B, redefining the V function based on the above updating rules completes the proof of our claim for this case. Further, the initial price of the European floating strike lookback Call option is obtained as,

$$C(S_0, m(t_0), t_0, \sigma(t_0)) = S_0 V(0, 0, t_0, \sigma(t_j)).$$

3 Extensions

One of the important things to take care while pricing a discrete exotic option on a lattice is the number of fixings of the contract. For example, if the lookback option in our case uses only weekly fixings to compute the minimum, then for pricing a six-month option on a lattice with 260 time periods, only the price at every tenth time period should be used for updating the running minimum. Instead if the minimum is updated at each of the 260 time periods, then the price thus obtained will not correspond to the contract with weekly fixings.

This issue of observation frequency is addressed next, using the standard modification available for the Cheuk-Vorst type algorithms. Among the time points denoted in the lattice, let $t_\ell, \ell = 0, Z, 2Z, \dots, LZ = N$ be the observation points. Thus between any two observation periods, there are $Z - 1$ time points. The example in the previous paragraph corresponds to $L = 26$ and $Z = 10$.

The only modification that is needed in our algorithm is a change in the updating rule when $t_j \neq iZ - 1$ ($i = 1, 2, \dots, L$). For $t_j \neq iZ - 1$, we will have,

$$(k_1, k_2, t_j, \sigma_L) \begin{cases} (k_1 + 1, k_2, t_j, \sigma_L/\sigma_H) \\ (k_1 - 1, k_2, t_j, \sigma_L/\sigma_H) \end{cases}$$

and

$$(k_1, k_2, t_j, \sigma_H) \begin{cases} (k_1, k_2 + 1, t_j, \sigma_L/\sigma_H) \\ (k_1, k_2 - 1, t_j, \sigma_L/\sigma_H) \end{cases}$$

In other words, for downward moves at these nodes we do not check whether a new minimum is reached, as these are non-observation points. And for all nodes at $t_j = iZ - 1$, we retain the usual updating rule as before. Defining the V function as per these updating rules will correctly price the lookback contracts with the given observation frequency. The final simplified algorithm incorporating all the three cases and the observation frequency, is given below.

Algorithm for pricing a Lookback Call option

```

-----
Read N, T, L, Z, rf, rd, S0, sig_L, sig_H, p_L, p_H
Assign the value for xi based on u_L and u_H
Define V at time N

for j = N-1 to 0
  if j != iZ-1 (i=1, ..., L)
    Use the values at (k1+1,k2) and (k1-1,k2) to update V for sig_L
    Use the values at (k1,k2+1) and (k1,k2-1) to update V for sig_H
  else
    Use the values at (k1+1,k2), subject to (k1+1)*xi + k2 > 0, and at (k1-1,k2),
    subject to (k1-1)*xi + k2 > 0, to update V for sig_L.
    If any of these condition is not met, use the values at (0,0) instead.
    Use the values at (k1,k2+1), subject to k1*xi + (k2+1) > 0, and at (k1,k2-1),
    subject to k1*xi + (k2-1) > 0, to update V for sig_H.
    If any of these condition is not met, use the values at (0,0) instead.
endfor

```

Output prices as $S_0 * V$ (at time 0) for sig_L and sig_H respectively.

The same algorithm will be used for pricing an American lookback option, by performing an additional check for early exercise at each node. This is because, the intrinsic value of the lookback call option at any node is given by,

$$S(t_j) - m(t_j) = S(t_j)(1 - u_L^{-k_1} u_H^{-k_2}).$$

Thus for pricing an American option, we check if $1 - u_L^{-k_1} u_H^{-k_2} > V(k_1, k_2, t_j, \sigma(t_j))$ and if this is true, the value of V is replaced with this expression. This modification gives us the price of American lookback options.

Following the original Cheuk and Vorst algorithm, the above pricing code can be easily modified for pricing European and American lookback puts, which will involve the running maximum of the exchange rate, and also for European fixed strike lookback Call and Put options. However because of homogeneity considerations, this cannot be extended for American fixed strike lookbacks.

4 Numerical Results

Some sample option prices computed for different values of the inputs, using the proposed algorithm, are provided in this section. Our goal is to verify whether these prices follow some of the common phenomenon observed in the case of vanilla options by the earlier studies. We also tested our code by checking that we get the prices of lookback from a standard CRR Binomial tree (i.e. with the constant volatility assumption) in the special case when $\sigma_L = \sigma_H$, independent of the values for p_L and p_H .

Table 1 gives the prices of both European and American floating strike lookback calls, with an expiry of six months, where the exchange rate moves between two volatility levels, $\sigma_L = 15\%$ and $\sigma_H = 40\%$. The persistence levels used here are $p_L = p_H = 0.75$, i.e. there is a 25% chance of transition to the other volatility level from the current state. These numbers also give us the impact of the initial volatility level at the inception of the option contract. Annual domestic risk-free rate of 3% and a foreign rate of 7% were assumed.

Table 1: European and American option prices from the algorithm, for a floating strike lookback call with increasing time periods (N) in the lattice and with $S_0 = 100$, $T = 0.5$, $r_f = 0.07$ and

$r_d = 0.03$. Prices with initial volatility states at the high (H) and the low (L) levels are given in separate columns.

N	European (L)	European (H)	American (L)	American (H)
25	12.45	13.14	12.75	13.49
50	13.14	13.48	13.45	13.81
100	13.59	13.76	13.91	14.02
200	13.90	13.98	14.23	14.31

These numbers do not take into account the observation frequency, and for each of the computed price, the code checks for a new maximum at every time period. As observed by previous authors, this leads to slow convergence and also the prices will form an increasing sequence, as more observation points will only lead to higher prices. And the increasing sequence of prices will eventually converge to the price of a continuous lookback option.

The above issue is sorted out once we take in to account the observation frequency, as discussed in section 3. Table 2 shows prices for the European lookback option starting at the low volatility level with different values of L and Z . For example $T = 6$ months, $Z = 25$ and $L = 26$ corresponds to a lookback option with weekly fixings, with 24 time periods in between any two fixings. The values for $Z = 1$ are similar to the values in Table 1 under the column European(L).

Table 2: Taking into account observation frequency with the same input values as before.

L	Z				
	1	10	25	50	50 ($\sigma_{avg}=0.22$)
1	4.19	6.60	7.08	7.22	5.09
2 (quarterly)	5.99	8.44	8.73	8.81	6.25
6 (monthly)	9.60	10.74	10.87	10.90	7.79
26 (weekly)	12.49	12.64	12.69	12.70	9.14

We observe the following:

- (i) We find that the convergence is a little faster once we keep the number of fixings constant (across rows). Also the speed of convergence improves for higher values of L , since the bottom rows have more overall time periods in the lattice when compared to the top ones.

- (ii) For table 2, the exchange rate starts at a volatility level of 15% and has a 25% chance of moving to a higher volatility level of 40%. Suppose we had ignored this volatility transition and had priced this option using the constant volatility assumption, the prices we would have obtained is shown in the last column of this table. Here we took an approximate average volatility of 22%(=0.75x15 + 0.25x40) for the entire time period. We find that the options can be significantly underpriced while assuming this constant volatility.

We give two other examples of the possible investigations offered by a stochastic volatility model, with an efficient pricing algorithm in hand. First in table 3, we illustrate the effects of persistence levels on the option prices, using a 100 period lattice. Here we look at a one year option contract and we start with the case where the fixings are half-yearly. We considered a low volatility level of 5% and a high volatility level of 30%. The first and the last rows are extreme situations corresponding to very high persistence and very low persistence. For American options given here, we check for early exercise only at observation points. The code can be easily modified in case we want to allow early exercise between the fixings.

Table 3: European and American lookback call option prices for varying persistence levels with $S_0 = 100$, $T = 1$, $r_f = 0.07$, $r_d = 0.03$, $\sigma_L = 0.05$ and $\sigma_H = 0.3$.

(a) Z=50 , L=2 (half-yearly)				
(p_L, p_H)	European (L)	European (H)	American (L)	American (H)
(0.99,0.99)	4.62	9.05	4.70	9.39
(0.95, 0.95)	7.12	8.03	7.30	8.30
(0.8, 0.8)	7.73	7.94	7.96	8.19
(0.01, 0.01)	7.89	7.92	8.13	8.17
(b) Z=25 , L=4 (quarterly)				
(0.99,0.99)	5.46	10.43	5.63	10.97
(0.95, 0.95)	8.25	9.23	8.57	9.67
(0.8, 0.8)	9.02	9.24	9.42	9.67
(0.01, 0.01)	9.23	9.26	9.65	9.69

As the numbers show, when we start in the low volatility state both American and European option prices increase as we move from higher to lower persistence levels. This occurs because at a

low persistence level, it is more easier for the exchange rate to migrate to the higher volatility state. Instead when we start with the high volatility state, this pattern reverses and the prices decrease as expected. Also note that even for a small change in the persistence levels from 99% to 95%, there is a significant change in the option prices. We repeat this exercise by increasing the observation frequency to quarterly on the 100 period lattice. We note that the differences created by the initial volatility levels remain almost the same, if we increase the observation frequency, both at high and low rates of transition.

Second, we give here the prices of fractional lookbacks, the payoffs of which were given in section 1. For these contracts the value function at expiry needs to be slightly modified, since when α is bigger than 1 for the Call option, it is sometimes possible for the final exchange rate to end up below $\alpha m(t_N)$. So we define

$$V(k_1, k_2, t_N, \sigma(t_N)) = \max(1 - u_L^{-k_1} u_H^{-k_2}, 0).$$

The fractional lookback option prices with one year expiry are presented in table 4, using $Z = 5$, $L = 26$, i.e. a 130 time period lattice where the observation frequency is once every two weeks. We have kept the low volatility level at 20% and the high volatility at 40%. All the prices reported here are starting at the low volatility state. Along with the American option prices, we also give for an easy reference, the percentage increase in the European option price, a buyer would need to bear to get the early exercise rights.

Table 4: Prices of fractional lookback call options with $T = 1$, $S_0 = 100$, $r_d = 0.03$, $r_f = 0.07$,

$Z = 5, L = 26, \sigma_L = 0.2, \sigma_H = 0.4.$				
$(p_L, p_H) = (0.9, 0.2)$		$(p_L, p_H) = (0.4, 0.2)$		
α	European(L)	American(L)	European(L)	American(L)
1.0	12.54	13.21 (5.3%)	16.64	17.35 (4.3%)
1.1	6.92	7.25 (4.8%)	10.94	11.37 (3.9%)
1.2	3.63	3.77 (3.9%)	7.03	7.26 (3.3%)
1.3	1.83	1.89 (3.3%)	4.44	4.57 (2.9%)

Table 4 also considers two cases of low and high volatility. The first case, where persistence levels in the low and high states are 90% and 20% respectively, corresponds to a situation where the exchange rate is mostly at the low volatility state but occasionally experiences bursts of high

volatile periods and after which it quickly returns back to the calmer state. We look at the same setting in the second case except that the short bursts of high volatility are more frequent (a higher chance of 60% of migrating to the high volatility state).

The above numbers also tell us that, with $\alpha = 1.1$ (i.e. 10% relaxation in m_t), there is an almost 50% reduction in the option premiums and similar reductions in the premiums for subsequent values of α . Thus as a cheaper alternative to the full lookback, the fractional option contract is effective in reducing the premiums, while still maintaining the basic character of the lookback contract.

5 Summary

Allowing for stochastic volatility results in realistic pricing of option contracts, but with considerable computational efforts. In this paper we have reduced the computational burden significantly for lookback options, by fully utilizing the structure of the model together with the homogenous property of the contracts. In fact our procedure has allowed us to price a lookback option on the stochastic volatility lattice, with almost the same computational complexity involved for pricing a vanilla Call option. As an application of our algorithm, we have looked at the efficiency of a fractional lookback contract in bringing down the premiums under stochastic volatility. In addition, with this model we are able to observe the impact of volatility persistence on these prices.

References

- Aingworth, D.D., S.R. Das, and R. Motwani (2006), “A simple approach for pricing equity options with markov switching state variables.” *Quantitative Finance*, 6(2), 95–105.
- Cheuk, T.H.F. and T.C.F. Vorst (1997), “Currency lookback options and observation frequency: a binomial approach.” *Journal of International Money and Finance*, 16(2), 173–187.
- Conze, A. and Viswanathan (1991), “Path dependent options: The case of lookback options.” *Journal of Finance*, 46(5), 1893–1907.
- Costabile, M., A. Leccadito, I. Massabo, and E. Russo (2014), “A reduced lattice model for option pricing under regime-switching.” *Review of Quantitative Finance and Accounting*, 42, 667–690.
- Cox, J.C., S.A. Ross, and M. Rubinstein (1979), “Option pricing: A simplified approach.” *Journal of Financial Economics*, 7, 229–263.

- Gerber, H.U. and E.S.W. Shiu (2003), “Pricing lookback options and dynamic guarantees.” *North American Actuarial Journal*, 7(1), 48–66.
- Heston, S. (1993), “A closed-form solution for options with stochastic volatility with applications to bond and currency options.” *Review of Financial Studies*, 6, 327–343.
- Heynen, R.C. and H.M. Kat (1997), “Lookback options.” In *Pricing and Applications in Exotic Options: The State of the Art* (C. Clewlow and C. Strickland, eds.), International Thomson, London.
- Hull, J. and A. White (1993), “Efficient procedures for valuing european and american path-dependent options.” *Journal of Derivatives*, 1, 21–31.
- Kimura, T. (2011), “American fractional lookback options: Valuation and premium decomposition.” *SIAM Journal of Applied Mathematics*, 71(2), 517–539.
- Lai, T.L. and T.W. Lim (2004), “Exercise regions and efficient valuation of american lookback options.” *Mathematical Finance*, 14(2), 249–269.
- Naik, V. (1993), “Option valuation and hedging strategies with jumps in the volatility of asset returns.” *Journal of Finance*, 48(5), 1969–1984.
- Tiong, S. (2000), “Valuing equity-indexed annuities.” *North American Actuarial Journal*, 4(4), 149–163.

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