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Time Truncated Acceptance Sampling Plans for Fréchet Model

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Abstract

In this paper, we develop acceptance sampling plan when the lifetime experiment is truncated at a pre-assigned time. The minimum sample size required to ensure a specified median life of the experimental unit is provided when the lifetimes of the units follow Fréchet distribution. The operating characteristic values of the sampling plans as well as the producer's risk are also presented. Examples are provided for illustrative purposes.

Keywords – Acceptance sampling, Failure rates, Fréchet distribution, Life test, Minimum sample size, Producer's risk

I. INTRODUCTION

Acceptance sampling plan (ASP) is a useful tool for ensuring quality in the field of statistical quality control. It is widely used when the testing is destructive, the cost of complete and thorough inspection is very high and/or it takes too much time. A random sample is selected from the lot and on the basis of information yielded by the sample, a decision is made regarding the accepting or rejecting the lot. It is served as a tool in which the consumer decides to accept or to reject a lot of products manufactured by the producer, based on the results of a random sample selected from the lot. The plan decides the minimum sample size required to draw from the large lot to achieve certain acceptance and non-acceptance criteria for the lot. So, an ASP consists of the number of units on test (n) and the acceptance number (c) such that if there are at most c failures out of n, the lot is accepted. For a given ASP, the consumer's and producer's risk are the probabilities that a bad lot is accepted and a good lot is rejected, respectively. Usually, with every acceptance sampling plan, the associated consumer's and producer's risks are also presented. For more details, one may refer to [1] and [2].

Any life testing experiment is carried out to obtain the lifetime (i.e., time to failure) of an item. In practice, the life test is terminated at a pre-fixed time, called truncation time (t), and the number of failures that occurred during the time period is recorded. In such an experiment, one may be interested to determine the probability that an experimental unit which has satisfactory performance during the time period t is classified as a non-defective unit. The acceptance sampling procedures can therefore be applied to life tests.

The standard approach to handle this problem is to assume a parametric model for the lifetime distribution as it provides insight into characteristics of failure times and hazard functions that may not be available with non-parametric methods and then to find the minimum sample size required to ensure a certain mean/median life (known as quality parameter) of the lifetime distribution of the items in the lot, when the experiment is stopped at a pre-determined time t. Extensive work has been done on the ASP, assuming different parametric forms of the lifetime. ASP based on truncated life tests for exponential distribution was first discussed in [3]. The results were extended for the Weibull distribution in [4]. References [5] and [6] provided extensive tables on ASP for gamma, normal and log-normal distributions. References [7], [8], [9], [10], [11], and [12] provide the time truncated ASP for half-logistics, log-logistics, Rayleigh, generalized Birnbaum-Saunders, generalized exponential, and generalized Weibull distributions respectively.

The Fréchet distribution was introduced by Fréchet [13] as one of the extreme value distribution. The maximum of a random sample after proper randomization can only converge in distribution to one of the three possible distributions, the Gumbel distribution, the Frèchet distribution or the Weibull distribution as discussed in [14]. In usual practice also, we try to provide the best component to the system and so lives of components may be assumed to follow a Frèchet distribution. The distribution has been used in modeling and analyzing several extreme events including reliability, accelerated life testing, earthquakes, wind speeds, floods, rain fall, sea currents, stock index, and so on. The detailed discussion on various applications of the Fréchet distribution can be found in [15], [16], [17], [18], [19], and the references therein.

In this paper, we present a methodology to find the minimum sample size necessary to ensure a specified median life based on Fréchet distribution when the life test is terminated at a pre-assigned time, t, and when the observed number of failures does not exceed a given acceptance number, *c*. The decision procedure is to accept a lot only if the specified median life can be established with a pre-assigned high probability P*, which provides protection to the consumer. The life test experiment gets terminated at the time at which (c + 1)st failure is observed or at time t, whichever is earlier. For a given acceptance sampling plan, a good lot might be rejected with a non-zero probability and that is known as the producer's risk. For different acceptance plans, we present the associated producer's risk also, based on the operating characteristic function values. In practice, instead of median life the consumer may prefer to characterize the quality based on some other percentile point (may be 75-th percentile point). Some examples have been discussed for illustrative purposes.

II. METHODOLOGY

A random variable X is said to follow Fréchet distribution [12] with parameters (β, λ), written as FR(β, λ), if the distribution function of X is given by

$$F_{GW}(x) = e^{-\left(\frac{x}{\lambda}\right)^{-\beta}} \; ; \; x > 0,; \; \lambda,\beta > 0 \quad (1)$$

where β is the shape parameters and λ is the scale parameter. The pth percentile point of FR(β, λ), say $\xi_p = F_{FR}^{-1}(p;\beta,\lambda)$ is given by

$$\xi_p = \lambda\left[ln\left(\frac{1}{p}\right)\right]^{-\frac{1}{\beta}}. \qquad (2)$$

Therefore, the median of FR(β, λ) becomes

$$\xi_m = \frac{\lambda}{(\ln 2)^{1/\beta}}, \qquad (3)$$

which eventually yields

$$\lambda_m^0 = \xi_m^0 \, (\ln 2)^{\frac{1}{\beta}} \qquad (4)$$

where ξ_m^0 is the assured or targeted median of the experiment and λ_m^0 is the corresponding value of the scale parameter λ for given $\beta = \beta_0$. It is obvious that for fixed β, $\xi_m \geq \xi_m^0 \iff \lambda_m \geq \lambda_m^0$. Now we develop the ASP for the FR distribution with known β_0 to ensure that the median lifetime of the items under study exceeds a pre-determined quality provided by the consumer say ξ_m^0, equivalently λ exceeds λ_m^0 with a minimum probability P*. As discussed before, usually the test terminates at a pre-specified time t and the number of failures during this time point are noted. Based on the number of failed items, a confidence limit (lower) on the median is formed. In the present ASP, the target median is accepted, if and only if the number of failures at the end of the time point t does not exceed *c*, the acceptance number. Naturally, if more than c failures already occurs before t, there is no point in continuing the test. In this case as soon as (*c*+1)st failure takes place before time point t, the test terminates with the decision not to accept the lot. Under these circumstances, one wants to obtain the smallest sample size, *n*, required to achieve these objectives.

The problem can be formulated as follows: given 0<P*<1, ξ_m^0, t and *c*, we are to find out the smallest *n* so that if the observed number of failures does not exceed *c*, it is ensured that $\xi_m \geq \xi_m^0$ with a minimum probability P* i.e. to obtain *n* such that the inequality

$$\sum_{i=0}^{c} \binom{n}{i} p^i (1-p)^{n-i} \leq 1 - P^* \qquad (5)$$

is satisfied where

$$p = F_{FR}(t;\beta) = e^{-\left(\frac{t}{\lambda}\right)^{-\beta}}. \qquad (6)$$

Note that p is monotonically increasing in $\frac{t}{\lambda_m}$ for fixed α and β, i.e. $F_{GW}\left(\frac{t}{\lambda}\right) \leq F_{GW}\left(\frac{t}{\lambda_m}\right)$ for $\lambda \geq \lambda_m^0$ or equivalently, $\xi_m \geq \xi_m^0$. The operating characteristic (OC) function of the sampling plan $\left(n, c, \frac{t}{\lambda_m^0}\right)$ provides the probability of accepting the lot. For the above ASP, this probability is given by

$$OC(p) = \sum_{i=0}^{c} \binom{n}{i} p^i (1-p)^{n-i} \text{ for } \lambda \geq \lambda_m^0.$$

On the other hand, the ratio of the true median and the assured median can be found out under the same ASP when the producer's risk (say, γ) which is the probability of rejection of the lot when $\lambda \geq \lambda_m^0$, is also given. The producer's risk is given by

$\gamma = \sum_{i=c+1}^{n} \binom{n}{i} p^{i}(1-p)^{n-i}$. For the given sampling plan, and for given γ, one can obtain the minimum value of $\frac{\lambda}{\lambda_m^0}$ or, equivalently $\frac{\xi}{\xi_m^0}$, for which

$$p = e^{-\left(\frac{t}{\lambda}\right)^{-\beta}} = e^{-\left(\frac{t}{\lambda_m^0} * \frac{\lambda_m^0}{\lambda}\right)^{-\beta}} \qquad (7)$$

satisfies the inequality

$$\sum_{i=c+1}^{n} \binom{n}{i} p^{i}(1-p)^{n-i} \leq \gamma.$$

III. RESULTS

A. Description of Tables

Given $\beta = \beta_0$ and assured median ξ_m^0, we obtain λ_m^0 from (4), calculate $\frac{t}{\lambda_m^0}$ for given t and then get *p* from (6). Substituting *p* in (5), we get minimum sample size, *n*, required to attain the assured median life ξ_m^0 with probability atleast P*. Fréchet distributed lifetime having $\beta = 0.5$ is considered for illustrative purposes throughout the paper. Table 1 provides the minimum value of *n* for which the present time truncated ASP is satisfied. We have considered P* = 0.90, 0.95 and $\frac{t}{\lambda_m^0}$= 0.628, 1.571, 2.356, 3.927, 4.712 as found in [5], [6], [8], [10], and [12] which will make the comparison easier. The Operating characteristic function values for the same plan for different values of P* and $\frac{\lambda}{\lambda_m^0}$ are presented in Table 2. Table 3 represents the minimum ratio of the true median life to the assured median life for the acceptance of a lot with the producer's risk 0.05. Let us describe the tables with the following illustrations.

In Table 1, when P* = 0.90, $\frac{t}{\lambda_m^0}$ =1.571, *c* = 2, the corresponding value of *n* is 10. It implies that out of 10 items, if 2 items fail before time point t, then a 90% upper confidence interval of λ will be $(\frac{t}{1.571}, \infty)$. In other words, if out of 10 items, less than or equal to 2 items fail before time point t, then we can accept the lot with probability 0.90 with the assurance that

$$\lambda \geq \frac{t}{1.571} \Leftrightarrow \xi_m \geq \frac{t}{1.571} \frac{1}{(\ln 2)^{\frac{1}{0.5}}} = 1.417\ t.$$

In Table 2, when P* = 0.90, $\frac{t}{\lambda_m^0}$ =1.571, c = 2, the corresponding operating characteristic function value is 0.9703 when $\frac{\lambda}{\lambda_m^0} = 10$. It implies, if one accepts the above time truncated ASP, i:e: the lot is accepted if out of 10 items, less than or equal to 2 items fail before time point t, then if $\lambda \geq \frac{10t}{1.571}$ or $\xi_m \geq 10$*1.417t, then the lot will be accepted with probability at least 0.596.

In Table 3, we obtain $\frac{\xi}{\xi_m^0}$ for the acceptance of a lot with the producer's risk 0.05. In this case for example, when the consumer's risk is 10%, i:e: P* = 0.90, c = 2, $\frac{t}{\lambda_m^0}$ =1.571, the tabular value of $\frac{\xi}{\xi_m^0}$ = 9.36. It implies if $\xi_m \geq 9.36*1.417t$, then with n = 10 (as obtained from Table 1) and c = 2, the lot will be rejected with probability less than or equal to 0.05.

B. Tables

Table1.

Minimum sample size necessary to assure that the median life exceeds a given value λ_m^0, with probability P* and the corresponding acceptance number, c, using binomial probabilities

		t/λ_m^0					
P*	*C*	0.628	1.571	2.356	3.141	3.972	4.712
0.90	0	7	4	4	3	3	3
	1	13	8	6	6	5	5
	2	18	10	8	8	7	7
	3	22	13	11	10	9	9
	4	27	16	14	12	11	11
	5	31	19	17	14	13	13
	6	35	22	19	16	15	15
	7	40	24	22	18	17	17
	8	44	27	23	20	19	18
	9	48	29	25	22	21	20
	10	52	32	27	24	23	22
0.95	0	10	6	5	4	4	4
	1	16	9	7	6	5	6
	2	21	12	10	9	8	8
	3	25	15	13	11	10	10
	4	32	18	17	14	13	12
	5	36	21	19	16	15	14
	6	40	24	21	18	17	16
	7	44	27	23	21	18	18
	8	48	30	25	22	20	20
	9	52	32	27	24	22	21
	10	57	34	29	26	24	23

Table 2.

OC values for the time truncated ASP $(n, c, \frac{t}{\lambda_m^0})$ for a given P* and $\frac{t}{\lambda_m^0}$, when c=2.

		λ/λ_m^0				
P*	n	t/λ_m^0	2	4	6	8
0.90	12	0.628	0.6731	0.9334	0.9848	0.9959
	7	1.571	0.5931	0.8472	0.9361	0.9709
	6	2.356	0.5485	0.7948	0.8977	0.9454
	5	3.141	0.5927	0.8044	0.8954	0.9400
0.95	14	0.628	0.6623	0.9225	0.9812	0.9945
	8	1.571	0.5868	0.8263	0.9311	0.9687
	7	2.356	0.5291	0.7884	0.8897	0.9365
	6	3.141	0.5374	0.7902	0.8934	0.9399

Table 3:

Minimum ratio of true median life to specified median life for the acceptance of a lot with producer's risk of 0.05

		t/λ_m^0				
P*	c	0.628	1.571	2.356	3.141	3.972
0.90	0	9.25	11.23	13.53	16.98	20.32
	1	8.32	10.54	11.06	12.67	14.55
	2	7.55	9.36	10.31	11.35	12.79
	3	6.30	8.59	9.35	10.11	11.38
	4	5.68	7.68	8.15	9.67	10.12
	5	5.23	6.74	7.44	8.21	9.60
	6	4.63	5.87	6.29	7.82	8.54
	7	3.25	4.21	5.89	6.42	7.06
	8	2.98	3.52	4.98	5.61	6.32
	9	2.31	3.36	4.27	5.11	6.14
	10	1.98	2.12	2.68	4.34	5.25
0.95	0	11.12	12.36	15.96	18.39	22.27
	1	9.06	10.14	11.98	13.12	16.31
	2	8.65	9.85	10.08	11.95	13.67
	3	7.69	8.12	9.65	10.17	12.11
	4	6.21	7.35	8.55	9.14	11.02
	5	5.74	6.87	7.81	8.14	9.99
	6	5.23	6.13	7.19	6.85	9.15
	7	4.37	5.62	6.54	7.18	8.69
	8	3.18	4.35	5.38	6.57	7.37
	9	2.38	3.91	4.19	5.95	6.54
	10	2.09	3.25	3.14	4.13	5.16

V. CONCLUSION

In any acceptance sampling plan, it is crucial to determine the minimum sample size required to draw from a large lot of items for the acceptance or rejection of the lot subject to attaining an assured average life of the items for given consumer’s risk. We have proposed such an ASP for Fréchet distribution here which has wide applications in extreme theory including reliability, finance, hydrology etc. We have considered the time truncated acceptance sampling plan in this paper for the Fréchet distribution. We have assumed that the shape parameter is known and presented the table for the minimum sample size required to assure a certain median life of the experimental units. We have also presented the operating characteristic function values and the associated producer's risks along with the minimum ratio of the true median life to the assured median life. The current plan is able to produce similar tables for the shape parameters and other percentiles also. As future research problems, one can think of applying accelerated life testing in ASP or using Bayesian ASP in the current life testing procedure.

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